# Analysis and Control of LPV Systems using Hybrid Systems Methods

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C·T·S·B
CONTROL THEORY & SYSTEMS BIOLOGY

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# Outline

1. (Short) introduction on LPV systems with some discussions
2. Stability analysis of a class of LPV systems with piecewise constant parameters
3. Control of a class of LPV systems with piecewise constant parameters
4. Hybrid systems formulation with some discussion

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# Introduction

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# LPV systems

## LPV systems

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t),\ x(0) = x_0 \tag{1}$$

where

- $x$ and $u$ are the state of the system and the control input
- $\rho(t) \in \mathcal{P}$, $\mathcal{P}$ compact, is the value of the parameter vector at time $t$
- the matrix-valued functions $A(\cdot)$ and $B(\cdot)$ are “nice enough", i.e. continuous

## Rationale

- Can be used to model a wide variety of real-world processes
- Convenient framework for the design gain-scheduled controllers/observers/filters

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# Quadratic stability

## Definition

*The LPV system*

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) \\ x(0) &= x_0 \end{aligned} \tag{2}$$

*is said to be quadratically stable if* $V(x) = x^\top P x$ *is a Lyapunov function for the system.*

## Theorem

*The LPV system* (2) *is quadratically stable if and only if there exists a matrix* $P \in \mathbb{S}^n_{\succ 0}$ *such that the LMI*

$$A(\theta)^\top P + PA(\theta) \prec 0 \tag{3}$$

*holds for all* $\theta \in \mathcal{P}$.

## Remarks

- All possible trajectories for the parameters are considered (with the restriction of existence of solutions)
- Semi-infinite dimensional LMI problem

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# Robust stability

## Definition

*The LPV system*

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) \\ x(0) &= x_0 \end{aligned} \tag{4}$$

*with* $\rho \in C^1(\mathbb{R}_{\geq 0}, \mathcal{P})$ *and* $\dot{\rho} \in \mathcal{D}$*, for some given compact sets* $\mathcal{P}, \mathcal{D} \subset \mathbb{R}^N$*, is said to be robustly stable if* $V(x, \rho) = x^\top P(\rho)x$ *is a Lyapunov function for the system.*

## Theorem

*The LPV system* (4) *is robustly stable if and only if there exists a differentiable matrix-valued function* $P : \mathcal{P} \to \mathbb{S}^n_{\succ 0}$ *such that the LMI*

$$\sum_{i=1}^{N} \theta_i' \partial_{\theta_i} P(\theta) + A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0 \tag{5}$$

*holds for all* $\theta \in \mathcal{P}$ *and all* $\theta' \in \mathcal{D}$*.*

## Remarks

- Trajectories of the parameters are continuously differentiable
- Infinite-dimensional LMI problem

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# Summary

## Some remarks

- Two main classes of parameter trajectories associated with two main stability concepts
- But these classes are very far apart!
- Parameter trajectories are defined in a quite loose way
- Part of the success of periodic, switched and jump systems lies in the “tailoredness” of the tools

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- Two main classes of parameter trajectories associated with two main stability concepts
- But these classes are very far apart!
- Parameter trajectories are defined in a quite loose way
- Part of the success of periodic, switched and jump systems lies in the "tailoredness" of the tools

## Issues

- What if we consider piecewise constant or piecewise differentiable parameters?
- Robust stability not applicable and quadratic stability too conservative
- Need something else!

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# Stability analysis of a class of LPV systems with piecewise constant parameters

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# LPV systems with PC parameters

## Two main classes of parameters

- Periodic jumps $\rightarrow$ constant dwell-time (time between two successive jumps)
- Aperiodic jumps $\rightarrow$ minimum dwell-time

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# LPV systems with PC parameters

## Two main classes of parameters

- Periodic jumps $\rightarrow$ constant dwell-time (time between two successive jumps)
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## Stability results

- Discrete-time-like stability conditions
- Conditions obtained using lifting ideas

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## Discussions

- Connections with quadratic and robust stability
- Example

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# Stability under constant dwell-time

Let us consider the LPV system

$$\dot{x} = A(\rho)x,\ x(0) = x_0 \tag{6}$$

with piecewise-constant parameter vector $\rho \in \mathscr{P}_T$ where

$$\mathscr{P}_T = \left\{ \begin{array}{c} \rho : \mathbb{R}_{\geq 0} \to \mathcal{P} :\ \rho(t) = \alpha_k \in \mathcal{P}, \\ t \in [t_k, t_{k+1}), t_k = kT,\ k \in \mathbb{N}_0 \end{array} \right\} \tag{7}$$

Defining then the discrete-time system

$$x(t_{k+1}) = e^{A(\alpha_k)T} x(t_k) \tag{8}$$

leads us to the following result:

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Defining then the discrete-time system

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leads us to the following result:

## Theorem

*Assume that there exists matrix-valued function $P : \mathcal{P} \to \mathbb{S}^n_{\succ 0}$ such that*

$$e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0 \tag{9}$$

*holds for all $\theta, \eta \in \mathcal{P}$.*
*Then, the LPV system is asymptotically stable for all $\rho \in \mathscr{P}_T$.*

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# Stability under minimum dwell-time

Let us consider the LPV system

$$\dot{x} = A(\rho)x,\ x(0) = x_0 \tag{10}$$

with piecewise-constant parameter vector $\rho \in \mathscr{P}_{\geqslant T}$ where

$$\mathscr{P}_{\geqslant T} = \left\{ \begin{array}{c} \rho : \mathbb{R}_{\geq 0} \to \mathcal{P} : \rho(t) = \alpha_k \in \mathcal{P}, \\ t \in [t_k, t_{k+1}), t_{k+1} \geq t_k + T, k \in \mathbb{N}_0 \end{array} \right\}. \tag{11}$$

The discrete-time system is given in this case by

$$x(t_{k+1}) = e^{A(\alpha_k)T_k} x(t_k), T_k \geq T \tag{12}$$

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*and*

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0 \tag{14}$$

*holds for all $\theta, \eta \in \mathcal{P}$.*
*Then, the LPV system is asymptotically stable for all $\rho \in \mathscr{P}_{\geqslant T}$.*

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# Issues

## Conditions

$$e^{A(\theta)^\top T} P(\theta) e^{A(\theta) T} - P(\eta) \prec 0, \quad A(\theta)^\top P(\theta) + P(\theta) A(\theta) \prec 0$$

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## Verification of the conditions

- Infinite-dimensional LMIs
- Exponential terms $e^{A(\theta)T}$ difficult to consider even when $A(\theta)$ is affine
- Gridding possible but inaccurate and computationally expensive

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- Infinite-dimensional LMIs
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- Gridding possible but inaccurate and computationally expensive

## Control design

- Nonconvex at all since we would have something like $e^{(A(\theta)+B(\theta)K(\theta))T}$

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# Lifted conditions - Constant DT

## Theorem

*The following statements are equivalent:*

(a) *There exists a matrix-valued function* $P : \mathcal{P} \to \mathbb{S}^n_{\succ 0}$ *such that the condition*

$$e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0 \tag{15}$$

*holds for all* $\theta, \eta \in \mathcal{P}$.

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*holds for all* $\theta, \eta \in \mathcal{P}$.

(b) *There exists a matrix-valued function* $S : [0, T] \times \mathcal{P} \to \mathbb{S}^n$, $S(T, \theta) \succ 0$, *such that the conditions*

$$\partial_\tau S(\tau, \theta) + S(\tau, \theta) A(\theta) + A(\theta)^T S(\tau, \theta) \preceq 0 \tag{16}$$

*and*

$$S(0, \theta) - S(T, \eta) \prec 0 \tag{17}$$

*hold for all* $\theta, \eta \in \mathcal{P}$ *and all* $\tau \in \mathcal{T} := [0, T]$.

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*hold for all* $\theta, \eta \in \mathcal{P}$ *and all* $\tau \in \mathcal{T} := [0, T]$.

*Moreover, when one of the above statements holds, then the LPV system is asymptotically stable for all* $\rho \in \mathscr{P}_T$.

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# Sketch of the proof

## (b) $\Rightarrow$ (a)

- Integrating $\partial_\tau S(\tau,\theta) + \mathrm{Sym}[S(\tau,\theta)A(\theta)] \preceq 0$ over $\tau \in [0,T]$ yields

$$e^{A(\theta)^\top T} S(T,\theta) e^{A(\theta)T} \preceq S(0,\theta).$$

- Using now $S(0,\theta) - S(T,\eta) \prec 0$ yields the condition

$$e^{A(\theta)^\top T} S(T,\theta) e^{A(\theta)T} - S(T,\eta) \prec 0.$$

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## (a) $\Rightarrow$ (b)

- Assume that there exists $P(\cdot) \succ 0$ such that $e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0$
- Pick $S^*(\tau,\theta) = e^{-A(\theta)^\top \tau} S^*(0,\theta) e^{-A(\theta)\tau}$, $S^*(T,\cdot) := P(\cdot) \succ 0$
- Then, we have that $\partial_\tau S^*(\tau,\theta) + \mathrm{Sym}[S^*(\tau,\theta)A(\theta)] = 0$
- Moreover, we have that

$$S^*(0,\theta) - S^*(T,\eta) = e^{A(\theta)^\top T} S^*(T,\theta) e^{A(\theta)T} - S^*(T,\eta) \prec 0 \qquad (18)$$

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# Lifted conditions - Minimum DT

## Theorem

*The following statements are equivalent:*

(a) *There exists a matrix-valued function* $P : \mathcal{P} \to \mathbb{S}^n_{\succ 0}$ *such that the conditions*

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0 \tag{19}$$

$$e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0 \tag{20}$$

*hold for all* $\theta, \eta \in \mathcal{P}$.

(b) *There exists a matrix-valued function* $S : [0, T] \times \mathcal{P} \to \mathbb{S}^n$, $S(T, \theta) \succ 0$, *such that the conditions*

$$A(\theta)^\top S(T, \theta) + S(T, \theta)A(\theta) \prec 0 \tag{21}$$

$$\partial_\tau S(\tau, \theta) + S(\tau, \theta)A(\theta) + A(\theta)^T S(\tau, \theta) \preceq 0 \tag{22}$$

$$S(0, \theta) - S(T, \eta) \prec 0 \tag{23}$$

*hold for all* $\theta, \eta \in \mathcal{P}$ *and all* $\tau \in \mathcal{T} := [0, T]$.

*Moreover, when one of the above statements holds, then the LPV system is asymptotically stable for all* $\rho \in \mathscr{P}_{\geqslant T}$.

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# Connection with quadratic and robust stability

## Theorem (Quadratic stability)

*When $T \to 0$ in the minimum dwell-time theorem, then the quadratic stability condition*

$$A(\theta)^\top P + PA(\theta) \prec 0 \tag{24}$$

*is recovered.*

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## Theorem (Robust stability)

*When $T \to \infty$ in the minimum dwell-time theorem, then the robust stability condition*

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0 \tag{25}$$

*for constant parametric uncertainties is recovered.*

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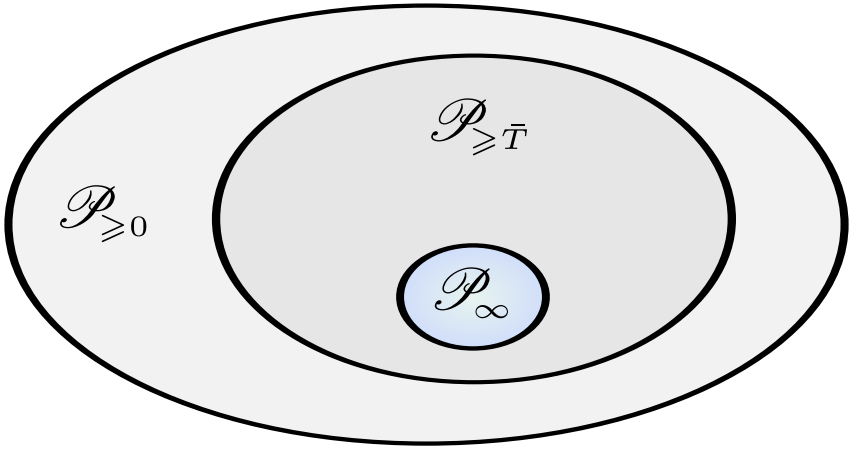

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# Example

## System

Let us consider the system (Wu, 1995, p. 55)

$$\dot{x} = \begin{bmatrix} 3/4 & 2 & \rho_1 & \rho_2 \\ 0 & 1/2 & -\rho_2 & \rho_1 \\ -3\upsilon\rho_1/4 & \upsilon\left(\rho_2 - 2\rho_1\right) & -\upsilon & 0 \\ -3\upsilon\rho_2/4 & \upsilon\left(\rho_1 - 2\rho_2\right) & 0 & -\upsilon \end{bmatrix} x \tag{26}$$

where $\upsilon = 15/4$ and $\rho \in \mathcal{P} = \{z \in \mathbb{R}^2 : ||z||_2 = 1\}$. This system is not quadratically stable.

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where $\upsilon = 15/4$ and $\rho \in \mathcal{P} = \{z \in \mathbb{R}^2 : ||z||_2 = 1\}$. This system is not quadratically stable.

## Results

- Gridding (126 points) of the exponential conditions yields the lower bound on the minimum dwell-time given by $T_\ell = 1.7594$ (complexity 256032/1260)
- Sum of squares conditions yield the upper-bounds on the minimum dwell-time of 2.7282 for polynomials of order 2 (complexity 7228/1350) and 1.7605 for polynomials of order 4 (complexity 33200/3500).

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# Control of a class of LPV systems with piecewise constant parameters

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# Setup

## System

- Let us consider the LPV system

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ x(0) &= x_0 \end{aligned}$$

where $\{t_k\}_{k\in\mathbb{N}_0}$ is the sequence of time instants at which the parameter vector changes value.

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where $\{t_k\}_{k\in\mathbb{N}_0}$ is the sequence of time instants at which the parameter vector changes value.

## Control laws

- Constant dwell-time case

$$u(t) = K(t - t_k, \rho(t_k))x(t),\ t \in [t_k, t_{k+1}) \tag{27}$$

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## Control laws

- Constant dwell-time case

$$u(t) = K(t - t_k, \rho(t_k))x(t),\ t \in [t_k, t_{k+1}) \tag{27}$$

- Minimum dwell-time case

$$u(t) = \begin{cases} K(t - t_k, \rho(t_k))x(t), & t \in [t_k, t_k + T) \\ K(T, \rho(t_k))x(t), & t \in [t_k + T, t_{k+1}) \end{cases} \tag{28}$$

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# State-feedback control - Minimum DT

## Theorem

*Assume that there exists matrix-valued functions* $U : [0, T] \times \mathcal{P} \to \mathbb{R}^{m \times n}$ *and* $\tilde{S} : [0, T] \times \mathcal{P} \to \mathbb{S}^n$, $\tilde{S}(T, \theta) \succ 0$, *such that the conditions*

$$\mathrm{Sym}[A(\theta)\tilde{S}(T, \theta) + B(\theta)U(T, \theta)] \prec 0, \tag{29}$$

$$-\partial_\tau \tilde{S}(\tau, \theta) + \mathrm{Sym}[A(\theta)\tilde{S}(\tau, \theta) + B(\theta)U(\tau, \theta)] \preceq 0 \tag{30}$$

*and*

$$\tilde{S}(T, \eta) - \tilde{S}(0, \theta) \prec 0 \tag{31}$$

*hold for all* $\theta, \eta \in \mathcal{P}$ *and all* $\tau \in [0, T]$.

*Then the closed-loop LPV system is asymptotically stable for all* $\rho \in \mathscr{P}_{\geqslant T}$, *and a suitable controller gain is moreover given by*

$$K(\tau, \theta) = U(\tau, \theta)\tilde{S}(\tau, \theta)^{-1}. \tag{32}$$

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# Example

## System

$$\dot{x} = \begin{bmatrix} 3-\theta & 1 \\ 1-\theta & 2+\theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1+\theta \end{bmatrix} u, \ \theta \in [0,1]. \tag{33}$$

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## System

$$\dot{x} = \begin{bmatrix} 3-\theta & 1 \\ 1-\theta & 2+\theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1+\theta \end{bmatrix} u, \ \theta \in [0,1]. \tag{33}$$

## Proposition

*No control law of the form $u = K(\theta)x$ can quadratically stabilize the system* (33)*.*

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## Proposition

*No control law of the form $u = K(\theta)x$ can quadratically stabilize the system* (33)*.*

## Proof

- Quadratically stabilizable if and only if the LMI (elimination lemma)

$$L(\theta) := B_\perp(\theta)[A(\theta)P + PA(\theta)^\top]B_\perp(\theta)^\top \prec 0$$

is feasible for all $\theta \in [0,1]$ where $B_\perp(\theta) = \begin{bmatrix} 1+\theta & -1 \end{bmatrix}$.

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  is feasible for all $\theta \in [0,1]$ where $B_\perp(\theta) = \begin{bmatrix} 1+\theta & -1 \end{bmatrix}$.
- Assume it is stabilizable, then $L(0) \prec 0$ and $L(1) \prec 0$.

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$$\dot{x} = \begin{bmatrix} 3-\theta & 1 \\ 1-\theta & 2+\theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1+\theta \end{bmatrix} u, \ \theta \in [0,1]. \tag{33}$$

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  is feasible for all $\theta \in [0,1]$ where $B_\perp(\theta) = \begin{bmatrix} 1+\theta & -1 \end{bmatrix}$.
- Assume it is stabilizable, then $L(0) \prec 0$ and $L(1) \prec 0$.
- This implies that there exists a $p \in \mathbb{R}$ such that

$$f_1(p) = p^2 - 3p + 2 < 0 \quad \text{and} \quad f_2(p) = p^2 - 6p + 8 < 0. \tag{34}$$

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$$L(\theta) := B_\perp(\theta)[A(\theta)P + PA(\theta)^\top]B_\perp(\theta)^\top \prec 0$$

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- Assume it is stabilizable, then $L(0) \prec 0$ and $L(1) \prec 0$.
- This implies that there exists a $p \in \mathbb{R}$ such that

$$f_1(p) = p^2 - 3p + 2 < 0 \quad \text{and} \quad f_2(p) = p^2 - 6p + 8 < 0. \tag{34}$$

- But $f_1(p) < 0 \Leftrightarrow p \in (1,2)$ and $f_2(p) < 0 \Leftrightarrow p \in (2,4)$, a contradiction.

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# Example

- We pick $T = 0.05$, polynomials $S, U$ of order 1

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- We pick $T = 0.05$, polynomials $S, U$ of order 1
- Primal/dual variables: 551/120; computation time: less than 2sec

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- We pick $T = 0.05$, polynomials $S, U$ of order 1
- Primal/dual variables: 551/120; computation time: less than 2sec
- We find

$$K(\tau,\theta) = \frac{1}{\text{den}(\tau,\theta)} \left[ \begin{array}{cc} K_1(\tau,\theta) & K_2(\tau,\theta) \end{array} \right]$$

where

$$\begin{array}{lcl} K_1(\tau,\theta) & = & 76.930 - 1109.596\tau + 14.343\theta + 1569.878\tau^2 + 170.469\tau\theta - 9.158\theta^2 \\ K_2(\tau,\theta) & = & 24.445 - 739.302\tau - 17.004\theta + 1136.874\tau^2 + 159.427\tau\theta + 3.174\theta^2 \\ \text{den}(\tau,\theta) & = & -23.189 + 483.241\tau - 0.934\theta - 947.359\tau^2 + 3.140\tau\theta + 1.066\theta^2 \end{array}$$

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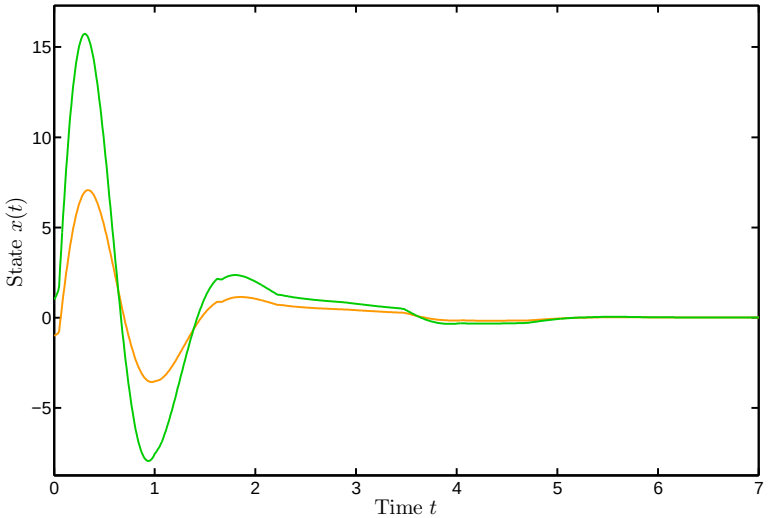


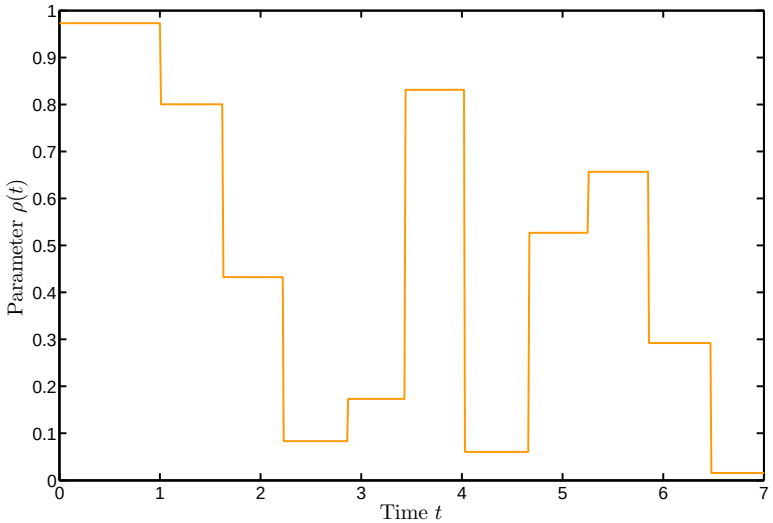

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# Hybrid systems formulation

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# Main problems

Is it possible to unify the frameworks of quadratic/robust stability and those of piecewise constant/differentiable parameters ?

Is it possible to generalize the approach to more complicated cases?

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# Stability analysis of hybrid systems

Let us consider the following hybrid inclusion

$$\begin{array}{lcl} \dot{x} & \in & F(x),\ x \in C \\ x^+ & \in & G(x),\ x \in D \end{array} \tag{35}$$

where $F$ and $G$ are set-valued maps and a closed set $\mathcal{A} \subset \mathbb{R}^n$.

## Theorem ([a])

*If the function $V : C \cup D \cup G(D) \to \mathbb{R}$ is a Lyapunov function candidate for the system and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function $\alpha_3$ such that*

(a) $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ *for all* $x \in C \cup D \cup G(D)$

(b) $\langle \nabla V(x), f \rangle \leq -\alpha_3(|x|_{\mathcal{A}})$ *for all* $x \in C$ *and* $f \in F$

(c) $V(g) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$ *for all* $x \in D$ *and* $g \in G(D)$

*then $\mathcal{A}$ is uniformly globally (pre-)asymptotically stable for the system.*

[a] R. Goebel, R. G. Sanfelice, and A. R. Teel, editors. *Hybrid Dynamical Systems. Modeling, Stability, and Robustness.* Princeton University Press, 2012

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*then $\mathcal{A}$ is uniformly globally (pre-)asymptotically stable for the system.*

## Relaxed conditions

- Persistent flowing $\to V(g) - V(x) \leq 0$ for all $x \in D$ and $g \in G(D)$
- Persistent jumping $\to \langle \nabla V(x), f \rangle \leq 0$ for all $x \in C$ and $f \in F$

[a] R. Goebel, R. G. Sanfelice, and A. R. Teel, editors. *Hybrid Dynamical Systems. Modeling, Stability, and Robustness.* Princeton University Press, 2012

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# Hybrid system formulation

## Hybrid form[a]

$$\begin{cases} \dot{x} &= A(\rho)x \\ \dot{\rho} &= 0 \\ \dot{\tau} &= 1 \end{cases} \quad \text{when} \quad \tau \in [0, T) \qquad (36)$$
$$\begin{cases} x^+ &= x \\ \rho^+ &\in \mathcal{P}\backslash\{\rho\} \\ \tau^+ &= 0 \end{cases} \quad \text{when} \quad \tau = T$$

- We augment the state with the parameter vector $\rho \in \mathcal{P}$ and a timer $\tau \in [0, 1]$
- State-space $\mathbb{R}^n \times \mathcal{P} \times [0, 1]$ where $\mathcal{P} := \mathcal{P}_1 \times \ldots \times \mathcal{P}_N$
- Stability to the set $\mathcal{A} = \{0\} \times \mathcal{P} \times [0, 1]$
- The jump part models parameters discontinuities
- The periodic case given above but the aperiodic case can be considered as well (e.g. $\dot{\tau} \in [0, 1)$)
- Using the previous theorem on this system, we recover stability conditions that are very similar to those previously obtained.

[a] R. Goebel, R. G. Sanfelice, and A. R. Teel, editors. *Hybrid Dynamical Systems. Modeling, Stability, and Robustness.* Princeton University Press, 2012

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# Deterministic parameters

## Multiple clocks

$$\begin{cases} \dot{x} &= A(\rho)x \\ \dot{\rho} &= 0 \\ \dot{\tau} &= 1 \end{cases} \quad \text{when} \quad \tau \in [0, T_1) \times [0, T_2) \times \ldots \times [0, T_N)$$

$$\begin{cases} x^+ &= x \\ \rho_i^+ &\in \mathcal{P}_i \backslash \{\rho_i\} \\ \rho_j^+ &= \rho_j, \; j \neq i \\ \tau_i^+ &= 0 \end{cases} \quad \text{when} \quad \tau_i = T_i, \; i = 1, \ldots, N \tag{37}$$

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## Piecewise differentiable parameters $\rho, \dot{\rho} \in [-1, 1]$

$$\begin{cases} \dot{x} &= A(\rho)x \\ \dot{\rho} &\in \left\{ \begin{array}{ll} [0,1] & \text{if } \rho = -1 \\ [-1,0] & \text{if } \rho = 1 \\ (-1,1) & \text{otherwise} \end{array} \right\} \\ \dot{\tau} &= 1 \end{cases} \quad \text{when} \quad \tau \in [0, T)$$

$$\left\{ \begin{array}{lcl} x^+ &=& x \\ \rho^+ &\in& [-1,1] \\ \tau^+ &=& 0 \end{array} \right\} \quad \text{when} \quad \tau = T \tag{38}$$

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# Stochastic parameters

## System (PDMP)

$$\begin{cases} \dot{x} &= A(\rho)x \\ \dot{\rho} &= 0 \end{cases} \quad \begin{cases} x^+ &= x \\ \rho^+ &= \theta \end{cases} \quad \text{with rate } \lambda(\rho,\theta) \tag{39}$$

- $\mathbb{P}[\rho(t+dt)=\eta|\rho(t)=\theta] = \lambda(\theta,\eta)dt + o(dt)$
- Random jump times (exponentially distributed dwell-times)
- Generalization of Markov jump linear systems

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## Proposition

*The above system is mean-square stable if (iff?) there exists a bounded matrix-valued function $P : \mathcal{P} \to \mathbb{S}^n_{\succ 0}$ such that the LMI*

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) + \int_{\mathcal{P}} \lambda(\theta,\eta)[P(\eta) - P(\theta)]d\eta \prec 0 \tag{40}$$

*holds for all $\rho \in \mathcal{P}$.*

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# Concluding remarks

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# Conclusions

## Concluding statements

- We can capture discontinuities in the parameters trajectories in a tractable way
- Extend quadratic and robust stability
- Extends results on switched systems
- Unification of the results seem to be possible using the framework of hybrid systems

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## What else can be done ?

- Performance analysis, e.g. $L_2$-performance
- Nonlinear systems (implicit computation of the state-transition operator)
- Homogeneous Lyapunov functions (on the basis of the converse results in[1])

[1] F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems, *SIAM Journal on Control and Optimization*, 2005

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## An open question

Is it possible to obtain tractable conditions for the design a dynamic output feedback?

[1] F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems, *SIAM Journal on Control and Optimization*, 2005

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# Thank you for your attention