

\mathcal{H}_∞ Bounded Resilient state-feedback design for linear continuous-time systems - A robust control approach

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- Introduction
- Main Results
- Examples

Introduction

- Resilience of controllers [Keel et al. '97]
 - Continuous-Time systems
 - Riccati [Haddad, 97], [Yang et al, 01]
 - LMI [Jadbabaie et al. 97], [Peaucelle et al. 04]
 - Discrete-Time Systems [Briat and Martinez, 09]
- Bounded controller design (NP-hard, [Blondel et al. 97])
 - Continuous-time [Peaucelle et al. 08]
 - Discrete-time [Briat and Martinez, 09]

Design of a state-feedback for continuous-time LTI systems

- Resilient (non-fragile)
- Bounded
- Achieve minimal performance

- LTI continuous-time linear systems

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ z(t) &= Cx(t) + Du(t) + Fw(t)\end{aligned}$$

state x , control input u , exogenous input w , controlled output z .

- Matrices supposed known
- Method easily extends to the uncertain case

- Find a control law of the form

$$u(t) = Kx(t)$$

which

- asymptotically stabilizes the system
- minimizes a performance criterion, e.g. \mathcal{H}_∞ .
- Additionally, the controller must satisfy
 - A resilience (non-fragility) property
 - A boundedness condition on the coefficients

Non-fragility property

- Self-robustness property of controllers
- Error on the controller implementation preserves closed-loop system stability
- Error models :
 - Additive error (rounding, uniform discrete valued space)

$$K_i = K_c + \delta K, \quad \delta K : \text{error term}$$

- Additive and multiplicative error (rounding+nonuniform discrete valued space)

$$K_i = K_c + \theta K_c + \Gamma, \quad \theta, \Gamma : \text{error term}$$

K_i implemented controller, K_c computed one

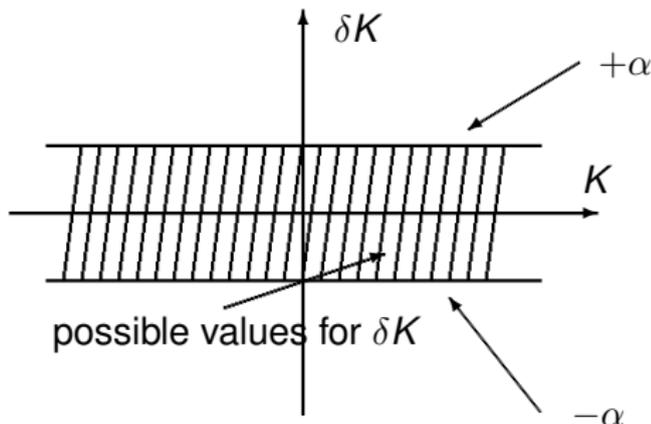
Additive Error

- Form of implemented gain

$$K_i = K_c + \delta K, \quad \delta K = U\Delta V$$

U, V known, Δ diagonal, $\|\Delta\|_2 \leq \alpha$

- Coefficients of δK inside $[-\alpha, \alpha]$



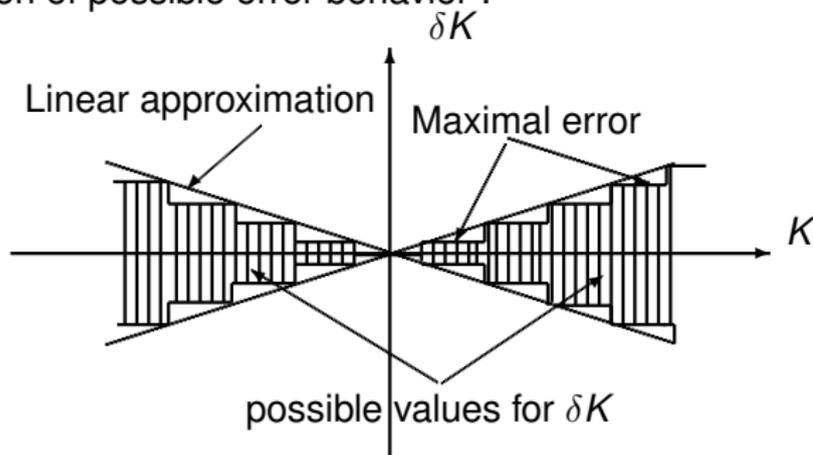
Additive-Multiplicative Error

- Form of implemented gain

$$K_i = (1 + \theta)K_c + \Gamma, \quad \Gamma = U\tilde{\Delta}V$$

$$U, V \text{ known, } \theta \in [-\mu, \mu], \quad \|\tilde{\Delta}\|_2 \leq \tilde{\alpha}$$

- Illustration of possible error behavior :



δK total implementation error

Bounded coefficients

- Form of implemented gain (with additive error)

$$K_i = \underbrace{K_0 + K_c}_{\text{previous } K_c} + \delta K$$

K_0 shifting term (K_c centered around 0)

- Design of a controller centered about 0 such that

$$\|K_c + \delta K\|_2 \leq \beta \sqrt{mn}$$

m, n dimensions of input and state resp, β maximal amplitude for controller coefficients.

Main Results

Resilient state-feedback (additive)

Theorem

There exists a quadratically stabilizing resilient state-feedback if there exist a matrix $X = X^T \succ 0$, a diagonal matrix $Q \succ 0$ and a scalar $\gamma > 0$ such that the following LMI

$$\begin{bmatrix} \mathcal{M}_{11} & E & XV^T & \mathcal{M}_{14} \\ * & -\gamma I & 0 & F^T \\ * & * & -Q & 0 \\ * & * & * & \mathcal{M}_{44} \end{bmatrix} \prec 0$$

holds where

$$\mathcal{M}_{11} = \text{He}[AX + BK_0X + BY] + \alpha^2 BUQU^T B^T$$

$$\mathcal{M}_{14} = [CX + DK_0X + DY + \alpha^2 DUQU^T B^T]^T$$

$$\mathcal{M}_{44} = -\gamma I + \alpha^2 DUQU^T D^T$$

In such a case, we have $K_c = YX^{-1}$ and the closed-loop system satisfies $\|z\|_{\ell_2} \leq \gamma \|w\|_{\ell_2}$.

Sketch of the proof

- Write the closed-loop system
- Substitute into the Bounded Real Lemma (BRL)
- Rewrite the BRL into the form

$$\Psi + U^T \Delta V + V^T \Delta^T U \prec 0$$

- Apply the Petersen's lemma (or Scaled-bounded real lemma), congruence transformations, Schur complement and change of variables (standard) to obtain LMIs

Adding constraints on the controller coefficients (1)

- Idea : Add a condition to the previous design
- Nonconvex constraint on the controller \rightarrow no exact LMI formulation
- Relaxation necessary (Cone complementary algorithm or iterative LMI algorithm)

Adding constraints on the controller coefficients (2)

- Iterative LMI based result (no additional optimization cost)

Theorem

Find N , Y and $X \succ 0$ of appropriate dimension such that

$$\begin{bmatrix} \Pi_{11} & Y & 0 & 0 \\ * & N^T X + X N & X V^T & N^T \\ * & * & -H & 0 \\ * & * & * & -I \end{bmatrix} \succ 0$$

$$\Pi_{11} = -mn\beta^2 I + \alpha^2 U H U^T$$

This will result in a gain K_c satisfying $\|K_c + \delta K\|_2 \leq \sqrt{mn}\beta$.

- Iteration between X and the slack-variable N
- Can be proved using the projection lemma.

Example

Example (1)

- Let us consider the unstable system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)$$

$$z(t) = Cx(t) + Du(t) + Fw(t)$$

with matrices $F = 0$ and

$$A = \begin{bmatrix} 2 & 1 & 7 & 1 \\ 1 & 7 & 1 & 2 \\ 7 & 1 & 0 & 1 \\ 2 & 1 & 6 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example (2)

- With no implementation error we get $\gamma^* = 3.4155$
- System stabilizable for all $\alpha < 1309$
- For a precision of $\alpha = 0.5$, we find $\gamma_a = 3.4368$ (worst case)
- Resilient Controller (after rounding)

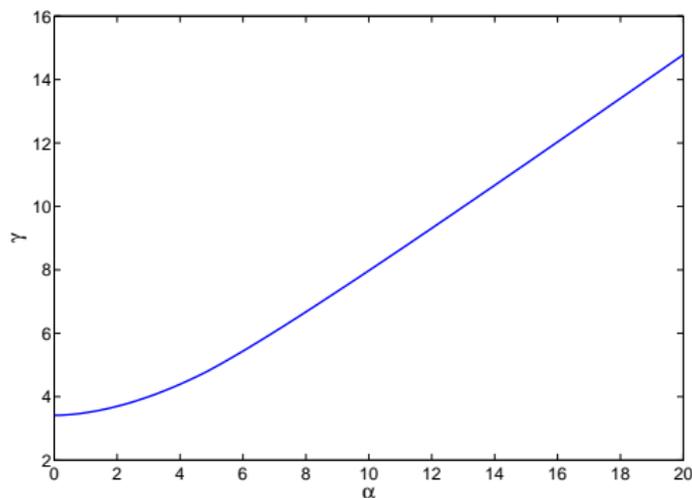
$$K_a = \begin{bmatrix} -4148 & -65751 & -10023 & -20577 \\ -22161 & -352006 & -53613 & -110142 \end{bmatrix}$$

- Too large coefficients
- Setting $\beta = 15.5$, $\alpha = 1/2$ and $K_0 = -1/2 \cdot \mathbb{1}_{m \times n}$ (integer coefficients in $[-16, 15]$), we get

$$K = \begin{bmatrix} -8 & 1 & -11 & -5 \\ -1 & -14 & 0 & -5 \end{bmatrix}$$

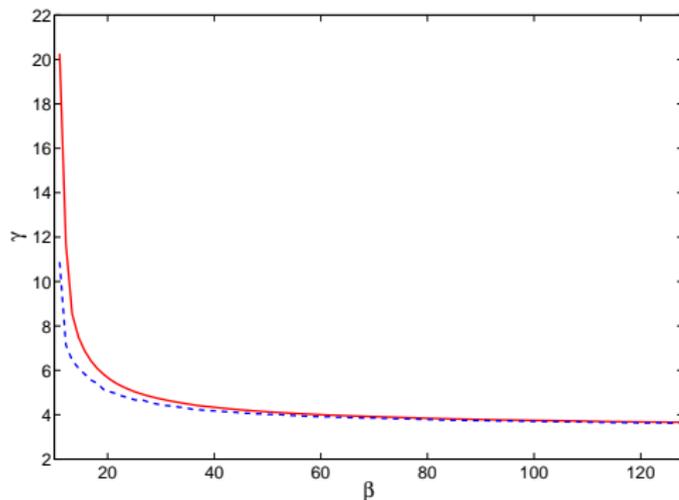
and $\gamma = 4.3924$.

Example (3)



Evolution of the \mathcal{H}_∞ -norm w.r.t. α

Example (4)



Evolution of the \mathcal{H}_∞ -norm w.r.t. β

- Red : worst-case (any controller in the ball of radius $\alpha = 1/2$)
- Blue : actual \mathcal{H}_∞ -norm for implemented controller

Conclusion and Future Works

Conclusion and Future Works

- Characterization of Resilient SF Controllers
- Two types of error
- LMI formulation (optimization)
- Additional nonlinear constraint for the boundedness of controllers (relaxation)

- Characterize more general class of errors
- Dynamic Output Feedback case
- Other formulations for boundedness of controllers

Thank you for your attention