ETH zürich

Simple interval observers for linear impulsive systems with applications to sampled-data and switched systems

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Motivation

- Impulsive systems are an important class of hybrid systems that can be used to represent switched and sampled-data systems [GST12]
- Linear positive impulsive systems have received less attention but some results exist [ZWXG14, Bri17]
- A particularity is that linear copositive Lyapunov functions $V(x) = \lambda^T x$, $\lambda > 0$ can be used to analyze them
- These results pave the way for the design of interval observers for impulsive systems [DER16]
- Interval observers are observers aiming to estimate upper and lower bounds on the state of a given system [GRH00]
- This talk is devoted to the design of such observers for linear impulsive systems and switched systems

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Linear impulsive systems

We consider here systems of the form

$$\begin{array}{rcl}\dot{x}(t_k+\tau) & = & A(\tau)x(t_k+\tau)+E_c(\tau)w_c(t_k+\tau),\ \tau\in(0,T_k]\\ x(t_k^+) & = & Jx(t_k)+E_dw_d(k),\ k\in\mathbb{N}\\ x(t_0) & = & x_0, t_0=0\end{array} \qquad (1)$$

where

- $x(t_k^+) := \lim_{s\downarrow t_k} x(s)$, $k\in\mathbb{N}$
- Timer variable τ measures the time elapsed since the last impulse/jump
- The sequence of impulse times $\{t_k\}_{k\in\mathbb{N}}$ is assumed to satisfy a minimum dwell-time constraint; i.e. $T_k := t_{k+1}-t_k \geq \bar{T}$ for all $k\in\mathbb{N}_0$

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Proposition

The following statements are equivalent:

(a) *The system* (1) *is state positive, i.e. for any* $x_0 \geq 0$, $w_c(t) \geq 0$ *and* $w_d(k) \geq 0$, *we have that* $x(t) \geq 0$ *for all* $t \geq 0$.

(b) *The matrix-valued function* $A(\tau)$ *is Metzler for all* $\tau \geq 0$, *the matrix-valued function* $E_c(\tau)$ *is nonnegative for all* $\tau \geq 0$ *and the matrices* J, E_d *are nonnegative.*

Stability under minimum dwell-time $\bar{T}$

Theorem ([Bri17])

Let us consider the system (1) with $w_c \equiv 0$, $w_d \equiv 0$, $A(\tau) = A(\bar{T})$ for all $\tau \geq \bar{T} > 0$, where $\bar{T} > 0$ is given, and assume that it is state positive. Then, the following statements are equivalent:

(a) *There exists a vector $\lambda \in \mathbb{R}^n_{>0}$ such that*

$$\lambda^T A(\bar{T}) < 0 \quad \text{and} \quad \lambda^T \left(\Phi(\bar{T})J - I_n\right) < 0 \tag{2}$$

hold where

$$\dot{\Phi}(s) = A(s)\Phi(s),\ \Phi(0) = I_n, s \in [0, \bar{T}]. \tag{3}$$

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hold where

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(b) *There exist a differentiable vector-valued* $\zeta : [0, \bar{T}] \mapsto \mathbb{R}^n$, $\zeta(\bar{T}) > 0$, *and a scalar* $\varepsilon > 0$ *such that the conditions*

$$\begin{array}{rcl} \zeta(\bar{T})^T A(\bar{T}) & < & 0 \\ -\dot{\zeta}(\tau)^T + \zeta(\tau)^T A(\tau) & \leq & 0 \\ \zeta(\bar{T})^T J - \zeta(0)^T + \varepsilon \mathbb{1}^T & \leq & 0 \end{array} \tag{4}$$

hold for all $\tau \in [0, \bar{T}]$.

Moreover, when one of the above statements holds, then the positive impulsive system (1) is asymptotically stable under minimum dwell-time $\bar{T}$.

Contents

The system

We consider here systems of the form

$$\begin{array}{rcl}\dot{x}(t) & = & Ax(t) + E_c w_c(t),\ t \notin \{t_k\}_{k\in\mathbb{N}} \\ x(t_k^+) & = & Jx(t_k) + E_d w_d(k),\ k \in \mathbb{N} \\ y_c(t) & = & C_c x(t) + F_c w_c(t),\ t \notin \{t_k\}_{k\in\mathbb{N}} \\ y_d(k) & = & C_d x(t_k) + F_d w_d(k),\ k \in \mathbb{N} \\ x(t_0) & = & x_0,\ t_0 = 0\end{array} \quad (5)$$

where

- $x, x_0 \in \mathbb{R}^n$, $w_c \in \mathbb{R}^{p_c}$, $w_d \in \mathbb{R}^{p_d}$, $y_c \in \mathbb{R}^{q_c}$ and $y_d \in \mathbb{R}^{q_d}$ are the state of the system, the initial condition, the continuous-time exogenous input, the discrete-time exogenous input, the continuous-time measured output and the discrete-time measured output, respectively.
- The sequence of impulse instants $\{t_k\}_{k\in\mathbb{N}}$ is assumed to satisfy the same properties as for the system (1).
- The input signals are all assumed to be bounded functions and that some bounds are known; i.e. we have $w_c^-(t) \le w_c(t) \le w_c^+(t)$ and $w_d^-(k) \le w_d(k) \le w_d^+(k)$ for all $t \ge 0$ and $k \ge 0$ and for some known $w_c^-(t), w_c^+(t), w_d^-(k), w_d^+(k)$.

The interval observer

We consider the following simple interval observer

$$
\begin{array}{rcl}
\dot{x}^\bullet(t) & = & Ax^\bullet(t) + E_c w_c^\bullet(t) + L_c(t)(y_c(t) - C_c x^\bullet(t) - F_c w_c^\bullet(t)) \\
x^\bullet(t_k^+) & = & Jx^\bullet(t_k) + E_d w_d^\bullet(t) + L_d(y_d(k) - C_d x^\bullet(t_k) - F_d w_d^\bullet(t)) \\
x^\bullet(t_0) & = & x_0^\bullet,\ t_0 = 0
\end{array}
\tag{6}
$$

where $\bullet \in \{-, +\}$

- The observer with the superscript "+"/"−" is meant to estimate an upper-bound/lower-bound; i.e. $x^-(t) \le x(t) \le x^+(t)$ for all $t \ge 0$ provided that $x_0^- \le x_0 \le x_0^+$, $w_c^-(t) \le w_c(t) \le w_c^+(t)$ and $w_d^-(k) \le w_d(k) \le w_d^+(k)$.

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The errors dynamics $e^+(t) := x^+(t) - x(t)$ and $e^-(t) := x(t) - x^-(t)$ are then described by

$$\begin{array}{rcl}\dot{e}^\bullet(t) & = & (A - L_c(t)C_c)e^\bullet(t) + (E_c - L_c(t)F_c)\delta_c^\bullet(t)\\ e^\bullet(t_k^+) & = & (J - L_dC_d)e^\bullet(t_k) + (E_d - L_dF_d)\delta_d^\bullet(k)\end{array} \quad (7)$$

where $\bullet \in \{-,+\}$, $\delta_c^+(t) := w_c^+(t) - w_c(t) \in \mathbb{R}^{p_c}_{\ge 0}$, $\delta_c^-(t) := w_c(t) - w_c^-(t) \in \mathbb{R}^{p_c}_{\ge 0}$, $\delta_d^+(k) := w_d^+(k) - w_d(k) \in \mathbb{R}^{p_d}_{\ge 0}$ and $\delta_d^-(k) := w_d(k) - w_d^-(k) \in \mathbb{R}^{p_d}_{\ge 0}$.

- Note that both errors have exactly the same dynamics → unnecessary here to consider different observer gains.

Interval observation problem - Minimum dwell-time

- In the minimum dwell-time case, the time-varying gain L_c is defined as follows

$$L_c(t_k+\tau) = \begin{cases} \tilde{L}_c(\tau) & \text{if } t \in (0,\bar{T}] \\ \tilde{L}_c(\bar{T}) & \text{if } t \in (\bar{T},T_k] \end{cases} \tag{8}$$

where $\tilde{L}_c : [0,\bar{T}] \mapsto \mathbb{R}^{n\times q_c}$ is a function to be determined.

- This structure is chosen to facilitate the derivation of convex design conditions.

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  where $\tilde{L}_c : [0,\bar{T}] \mapsto \mathbb{R}^{n\times q_c}$ is a function to be determined.
- This structure is chosen to facilitate the derivation of convex design conditions.
- The observation problem is defined as follows:

Problem

Find an interval observer of the form (6) *(i.e. a matrix-valued function* $L_c(\cdot)$ *of the form* (8) *and a matrix* $L_d \in \mathbb{R}^{n\times q_d}$*) such that the error dynamics* (7) *is*

(a) *state-positive, that is*
  - $A - L_c(\tau)C_c$ *is Metzler for all* $\tau \in [0,\bar{T}]$,
  - $E_c - L_c(\tau)F_c$ *is nonnegative for all* $\tau \in [0,\bar{T}]$,
  - $J - L_dC_d$ *and* $E_d - L_dF_d$ *are nonnegative; and*

(b) *asymptotically stable under minimum dwell-time* $\bar{T}$ *when* $w_c \equiv 0$ *and* $w_d \equiv 0$.

Theorem (Minimum dwell-time)

Assume that there exist a differentiable matrix-valued function $X : [0, \bar{T}] \mapsto \mathbb{D}^n$, $X(\bar{T}) \succ 0$, *a matrix-valued function* $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{n \times q_c}$, *a matrix* $U_d \in \mathbb{R}^{n \times q_d}$ *and scalars* $\varepsilon, \alpha > 0$ *such that the conditions*

$$X(\tau)A - U_c(\tau)C_c + \alpha I_n \geq 0 \tag{9a}$$

$$X(\bar{T})J - U_d C_d \geq 0 \tag{9b}$$

$$X(\tau)E_c - U_c(\tau)F_c \geq 0 \tag{9c}$$

$$X(\bar{T})E_d - U_d F_d \geq 0 \tag{9d}$$

and

$$\mathbb{1}_n^T \left[X(\bar{T})A - U_c(\bar{T})C_c + \varepsilon I_n\right] \leq 0 \tag{10a}$$

$$\mathbb{1}_n^T \left[-\dot{X}(\tau) + X(\tau)A - U_c(\tau)C_c\right] \leq 0 \tag{10b}$$

$$\mathbb{1}_n^T \left[X(\bar{T})J - U_d C_d - X(0) + \varepsilon I\right] \leq 0 \tag{10c}$$

hold for all $\tau \in [0, \bar{T}]$*. Then, there exists an interval observer of the form* (6)-(8) *that solves the interval observation problem and suitable observer gains are given by*

$$\tilde{L}_c(\tau) = X(\tau)^{-1}U_c(\tau) \quad \text{and} \quad L_d = X(\bar{T})^{-1}U_d. \tag{11}$$

Contents

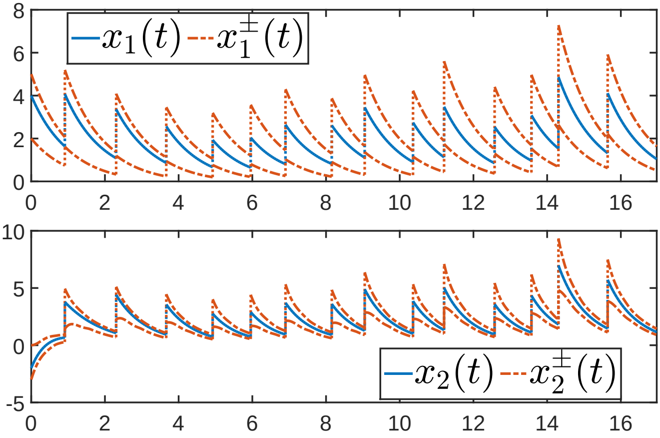
An impulsive system

- Let us consider here the example from [Bri13] to which we add disturbances as also done in [DER16]. The matrices of the system are given by

$$\begin{array}{rcl}\dot{x}(t) & = & \begin{bmatrix}-1 & 0\\ 1 & -2\end{bmatrix}x(t)+\begin{bmatrix}0.1\\ 0.1\end{bmatrix}w_c(t)\\ x(t_k^+) & = & \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}x(t_k)+\begin{bmatrix}0.3\\ 0.3\end{bmatrix}w_d(k)\\ y_c(t) & = & \begin{bmatrix}0 & 1\end{bmatrix}x(t)+0.03w_c(t)\\ y_d(k) & = & \begin{bmatrix}0 & 1\end{bmatrix}x(t_k)+0.03w_d(k)\end{array} \tag{12}$$

- We consider $w_c(t)=\sin(t)$, $w^-(t)=-1$, $w^+(t)=1$, $w_d(k)$ is a stationary random process that follows the uniform distribution $\mathcal{U}(-0.5,0.5)$, $w_d^-=-0.5$ and $w_d^+=0.5$.
- We choose a desired minimum dwell-time of $\bar{T}=0.7$ and solve the conditions of the theorem with polynomials of degree 4 (SOS method)
- The semidefinite program has 242 primal variables, 76 dual variables and it takes 2.18 seconds to solve on an i7-2620M with 8GB of RAM.

Simulation results



Switched system

- Let us consider here the switched system

$$\begin{array}{rcl}\dot{\tilde{x}}(t) & = & \tilde{A}_{\sigma(t)}\tilde{x}(t) + \tilde{E}_{\sigma(t)}w(t) \\ \tilde{y}(t) & = & \tilde{C}_{\sigma(t)}\tilde{x}(t) + \tilde{F}_{\sigma(t)}w(t)\end{array} \quad (13)$$

where $\sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \ldots, N\}$ is the switching signal, $\tilde{x} \in \mathbb{R}^n$ is the state of the system, $\tilde{w} \in \mathbb{R}^p$ is the exogenous input and $\tilde{y} \in \mathbb{R}^p$ is the measured output.

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- This system can be rewritten into the following impulsive system with multiple jump maps

$$\begin{array}{rcl}\dot{x}(t) & = & \mathrm{diag}_{i=1}^N(\tilde{A}_i)x(t)+\mathrm{col}_{i=1}^N(\tilde{E}_i)w(t)\\ y(t) & = & \mathrm{diag}_{i=1}^N(\tilde{C}_i)x(t)+\mathrm{col}_{i=1}^N(\tilde{F}_i)w(t)\\ x(t_k^+) & = & J_{ij}x(t_k),\ i,j=1,\ldots,N,\ i\neq j\end{array} \tag{14}$$

where $J_{ij}:=(b_ib_j^T)\otimes I_n$ and $\{b_1,\ldots,b_N\}$ is the standard basis for $\mathbb{R}^N$.

Switched system

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$$\begin{array}{rcl} \dot{\tilde{x}}(t) & = & \tilde{A}_{\sigma(t)}\tilde{x}(t) + \tilde{E}_{\sigma(t)}w(t) \\ \tilde{y}(t) & = & \tilde{C}_{\sigma(t)}\tilde{x}(t) + \tilde{F}_{\sigma(t)}w(t) \end{array} \tag{13}$$

where $\sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \ldots, N\}$ is the switching signal, $\tilde{x} \in \mathbb{R}^n$ is the state of the system, $\tilde{w} \in \mathbb{R}^p$ is the exogenous input and $\tilde{y} \in \mathbb{R}^p$ is the measured output.

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where $J_{ij} := (b_i b_j^T) \otimes I_n$ and $\{b_1, \ldots, b_N\}$ is the standard basis for $\mathbb{R}^N$.

- Because of the particular structure of the system, we can define an interval observer with the gains $L_c(t) = \mathrm{diag}_{i=1}^N(L_c^i(t))$ and $L_d^{ij} = (b_i b_j^T) \otimes \tilde{L}_d^{ij}$. The error dynamics is then given in this case by

$$\begin{array}{rcl} \dot{e}^\bullet(t) & = & \mathrm{diag}_{i=1}^N(\tilde{A}_i - L_c^i(t)\tilde{C}_i)e^\bullet(t) + \mathrm{col}_{i=1}^N(\tilde{E}_i - L_c^i(t)\tilde{F}_i)\delta^\bullet(t) \\ e^\bullet(t_k^+) & = & \left[(b_i b_j^T) \otimes I_n - \tilde{L}_d^{ij}\tilde{C}_j)\right] e^\bullet(t_k) - \left[(b_i b_j^T) \otimes (\tilde{L}_d^{ij}\tilde{F}_j)\right] \delta^\bullet(t_k). \end{array} \tag{15}$$

Contents

Conclusion

- Simple conditions for the design of interval observers
- Infinite dimensional LP conditions solved using SOS programming (so SDP in the end)
- The results can be improved by considering more complex observers, change of variables, sign decomposition, etc.
- Extensions to account for performance (e.g. L_1 performance) are possible
- Other dwell-time constraints s.t. maximum DT, average DT, etc.
- Unclear whether accurate results can also be obtained for constant observer gains but this is possible using certain polynomial approaches
- Use of interval observers for stabilization

Thank you for your attention
Any questions?

Computational aspects

Proposition

Let $d \in \mathbb{N}$, $\varepsilon > 0$ and $\epsilon > 0$ be given and assume that there exist polynomials $\chi_i : \mathbb{R} \mapsto \mathbb{R}$, $i = 1, \ldots, n$, $U_c : \mathbb{R} \mapsto \mathbb{R}^{n \times q_c}$, $\Gamma_1 : \mathbb{R} \mapsto \mathbb{R}^{n \times n}$, $\Gamma_2 : \mathbb{R} \mapsto \mathbb{R}^{n \times q_c}$ and $\gamma_1, \gamma_2 : \mathbb{R} \mapsto \mathbb{R}^n$ of degree $2d$, a matrix $U_d \in \mathbb{R}^{n \times q_d}$ and a scalar $\alpha \geq 0$ such that

(a) $\Gamma_i(\tau)$, $\gamma_i(\tau)$, $i = 1, 2$ *are componentwise-SOS (CSOS),*

(b) $X(0) - \epsilon I_n \geq 0$ *(or is CSOS),*

(c) $X(\tau)A - U_c(\tau)C_c + \alpha I_n - \Gamma_1(\tau)f(\tau)$ *is CSOS,*

(d) $X(0)J - U_d C_d \geq 0$ *(or is CSOS),*

(e) $X(\tau)E_c - U_c(\tau)F_c - \Gamma_2(\tau)f(\tau)$ *is CSOS,*

(f) $X(0)E_d - U_d F_d \geq 0$ *(or is CSOS),*

(g) $-\mathbb{1}_n^T \left[X(\bar{T})A - U_c(\bar{T})C_c + \varepsilon I_n\right]$ *is CSOS*

(h) $-\mathbb{1}_n^T \left[-\dot{X}(\tau) + X(\tau)A - U_c(\tau)C_c\right] - f(\tau)\gamma_1(\tau)^T$ *is CSOS,*

(i) $-\mathbb{1}_n^T \left[X(\bar{T})J - U_d C_d - X(0) + \varepsilon I\right]$ *is CSOS,*

where $X(\tau) := \mathrm{diag}_{i=1}^n(\chi_i(\tau))$, $f(\tau) := \tau(\bar{T} - \tau)$.
Then, the conditions of statement of the theorem hold with the same $X(\tau)$, $U_c(\tau)$, U_d, α *and* ε.

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