$\mathcal{H}_\infty$ Filtering of Uncertain LPV Systems with Time-Delay

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Outline

- Introduction
- Stability of Uncertain LPV Systems with Delays
- The filtering Problem
- Conclusion and Future Works

Introduction

- Considered Systems
- Filters Structures

Considered Systems (1)

- Uncertain LPV Systems with Delay

$$\begin{array}{rcl}
\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} & = & \Sigma(\rho(t),\delta) \begin{bmatrix} x(t) \\ x(t-h(t)) \\ w(t) \end{bmatrix} \\
x(\theta) & = & \phi(\theta),\ \theta \in [-h_M, 0] \\
\rho & \in & U_\rho \\
\dot{\rho} & \in & \mathrm{hull}[U_\nu] \\
\delta & \in & U_\delta \\
h(t) & \in & [0, h_M] \\
\dot{h}(t) & < & \mu < 1
\end{array}$$

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Considered Systems (2)

- System Matrix :

$$\Sigma = \begin{bmatrix} A(\rho) + \Delta A(\rho,\delta) & A_h(\rho) + \Delta A_h(\rho,\delta) & E(\rho) + \Delta E(\rho,\delta) \\ C(\rho) & C_h(\rho) & F(\rho) \\ C_y(\rho) + \Delta C_y(\rho,\delta) & C_{yh}(\rho) + \Delta C_{yh}(\rho,\delta) & F_y(\rho) + \Delta F_y(\rho,\delta) \end{bmatrix}$$

- where the uncertain part obeys

$$\begin{bmatrix} \Delta A & \Delta A_h & \Delta E \\ \Delta C_y & \Delta C_{yh} & \Delta F_y \end{bmatrix}(\rho,\delta) = \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}(\rho)\Delta(\delta)\begin{bmatrix} F_0 & F_1 & F_2 \\ F_3 & F_4 & F_5 \end{bmatrix}(\rho)$$

with $||\Delta||_2 \leq 1$

Filters Structures

- Filters with memory

$$\begin{bmatrix} \dot{x}_F(t) \\ z_F(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho) & A_{hF}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{hF}(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} x_F(t) \\ x_F(t-h(t)) \\ y(t) \end{bmatrix}$$

- Memoryless filters

$$\begin{bmatrix} \dot{x}_F(t) \\ z_F(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} x_F(t) \\ y(t) \end{bmatrix}$$

- These matrices are aimed to be chosen such that

$$||z - z_F||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

with a minimal $\mathcal{L}_2$-gain $\gamma > 0$.

Stability of Uncertain LPV Systems with Delays

- Lyapunov-Krasovskii Functional
- Asymptotic Stability Theorem
- Relaxed Version

Lyapunov-Krasovskii Functional (1)

- Delay-Dependent LKF :

$$\begin{array}{rcl} V(x_t,\dot{x}_t,\rho) & = & V_1(x_t,\rho)+V_2(x_t)+V_3(\dot{x}_t) \\ V_1(x_t,\rho) & = & x(t)^T P(\rho)x(t) \\ V_2(x_t) & = & \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta \\ V_3(\dot{x}_t) & = & h_M \displaystyle\int_{-h_M}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta \end{array}$$

- whose derivative along the trajectories solution of the system satisfies :

$$\begin{array}{rcl} \dot{V}_1 & = & \dot{x}(t)^T P(\rho)x(t)+x(t)^T P(\rho)\dot{x}(t)+x(t)^T\left(\displaystyle\sum_i \dot{\rho}_i \frac{\partial}{\partial \rho_i}P(\rho)\right)x(t) \\ \dot{V}_2 & \leq & x(t)^T Q x(t)-(1-\mu)x(t-h(t))^T Q x(t-h(t)) \\ \dot{V}_3 & \leq & h_M^2 \dot{x}(t)^T R \dot{x}(t)-h_M\displaystyle\int_{t-h(t)}^{t} \dot{x}(\theta)^T R \dot{x}(\theta) d\theta \end{array}$$

Lyapunov-Krasovskii Functional (2)

- Using Jensen's Inequality :

$$-h_M \int_{t-h(t)}^{t} \dot{x}(\theta)^T R \dot{x}(\theta) d\theta \quad \leq \quad -\left(\int_{t-h(t)}^{t} \dot{x}(\theta) d\theta\right)^T R \left(\int_{t-h(t)}^{t} \dot{x}(\theta) d\theta\right)$$

- we get

$$\dot{V} \leq \chi(t)^T \begin{bmatrix} \Psi + h_M^2 A^T R A & PA_h + R + h_M^2 A^T R A_h \\ \star & -(1-\mu)Q - R + h_M^2 A_h^T R A_h \end{bmatrix} \chi(t)$$

with $\Psi = A^T P + PA + Q - R + \sum_i \dot{\rho}_i \dfrac{\partial}{\partial \rho_i} P(\rho)$ and
$\chi(t) = col(x(t), x(t-h(t)))$.

Asymptotic Stability

Theorem

The unertain LPV system is asymptotically stable for all $h(t) \in [0, h_M]$ *such that* $\dot{h}(t) < \mu$ *if there exist* $P : U_\rho \to \mathbb{S}^n_{++}$, $Q, R \in \mathbb{S}^n_{++}$ *such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) & P(\rho)A_h(\rho) + R & h_M A(\rho)^T R \\ \star & -(1-\mu)Q - R & h_M A_h(\rho)^T R \\ \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *with*

$$\Psi(\rho,\nu) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \sum_i \nu_i \frac{\partial}{\partial \rho_i} P(\rho).$$

Bounded Real Lemma

Theorem

The unertain LPV system is asymptotically stable for all $h(t) \in [0, h_M]$ such that $\dot{h}(t) < \mu$ if there exist $P : U_\rho \to \mathbb{S}^n_{++}$, $Q, R \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that the LMI

$$\begin{bmatrix} \Psi(\rho,\nu) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) & C(\rho)^T & h_M A(\rho)^T R \\ \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & h_M A_h(\rho)^T R \\ \star & \star & -\gamma I & F(\rho)^T & h_M E(\rho)^T R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ with
$\Psi(\rho,\nu) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \sum_i \nu_i \dfrac{\partial}{\partial \rho_i} P(\rho)$. *Moreover, we have $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.*

Relaxed Version

Theorem

The unertain LPV system is asymptotically stable for all $h(t) \in [0, h_M]$ such that $\dot{h}(t) < \mu$ if there exist $P : U_\rho \to \mathbb{S}^n_{++}$, $X : U_\rho \to \mathbb{R}^{n \times n}$, $Q, R \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that the LMI

$$\begin{bmatrix} -(X+X^T) & P + X^T A & X^T A_h & X^T E & 0 & X^T & h_M R \\ \star & \Phi_1 & R & 0 & C^T & 0 & 0 \\ \star & \star & \Phi_2 & 0 & C_h^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P & -h_M R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ with $\Phi_1 = P + Q - R + \sum_i \nu_i \frac{\partial}{\partial \rho_i} P(\rho)$ and $\Phi_2 = -(1-\mu)Q - R$. Moreover, we have $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Filtering Problem

- Augmented System
- Relaxation
- Example

Augmented System

- Interconnection between the system and filter

$$
\begin{aligned}
\dot{x}_a(t) &= \mathcal{A}x_a(t) + \mathcal{A}_h x_a(t-h(t)) + \mathcal{E}w(t)\\
z_e(t) &= \mathcal{C}x_a(t) + \mathcal{C}_h x_a(t-h(t)) + \mathcal{F}w(t)\\
x_a(t) &= col(x(t), x_F(t))\\
z_e(t) &= z(t) - z_F(t)
\end{aligned}
$$

- with

$$
\mathcal{A} = \begin{bmatrix} A & 0 \\ A - B_F C_y & A_F \end{bmatrix} \quad \mathcal{A}_h = \begin{bmatrix} A_h & 0 \\ A_h - B_F C_{yh} & A_{Fh} \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} E \\ E - B_F F_y \end{bmatrix}
$$

$$
\mathcal{C} = \begin{bmatrix} C - D_F C_y - C_F & C_F \end{bmatrix} \quad \mathcal{C}_h = \begin{bmatrix} C_h - D_F C_{yh} - C_{Fh} & C_{Fh} \end{bmatrix}
$$

$$
\mathcal{F} = F - D_F F_y
$$

Relaxation of Bilinear Terms

- Bilinear Terms $X^T\mathcal{A}, X^T\mathcal{A}_h, X^T\mathcal{E}$

$$\begin{aligned} X^T\mathcal{A} &= \begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix} \begin{bmatrix} A & 0 \\ A - B_F C_y & A_F \end{bmatrix} \\ &= \begin{bmatrix} (X_1 + X_3)^T A - X_3^T B_F C_y & X_3^T A_F \\ (X_2 + X_4)^T A - X_4^T B_F C_y & X_4^T A_F \end{bmatrix} \end{aligned}$$

- Set $X_4 = X_3$ (both system and filter have the same order)
- Linearization

$$\begin{bmatrix} \tilde{A}_F & \tilde{A}_{hF} & \tilde{B}_F \end{bmatrix} = X_3^T \begin{bmatrix} A_F & A_{hF} & B_F \end{bmatrix}$$

- we get a LMI problem

Example 1

- Let

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}x(t) + \begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}x(t-h(t)) + \begin{bmatrix}0\\1\end{bmatrix}w(t)\\ z(t) &= \begin{bmatrix}1 & 2\end{bmatrix}x(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t)\end{aligned}$$

- We set $h_M = 1$ and we study γ w.r.t. μ using a memoryless filter

μ	0	0.4	0.8
Fridman [2003]	1.4086	1.8311	15.8414
This result	0.06484	0.10651	0.48661

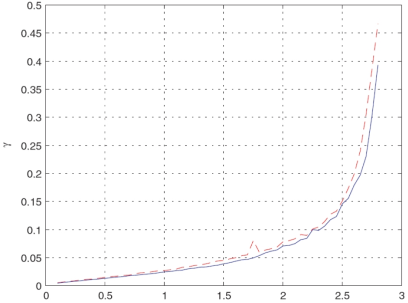
Example 2(1)

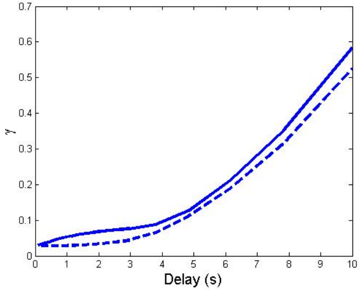
- We consider the LPV system

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x(t-h(t)) \\
&\quad + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\
z(t) &= \begin{bmatrix} 0.3 & 1.5 \\ -0.45 & 0.75 \end{bmatrix} x(t) + \begin{bmatrix} 0.5\rho \\ -0.5\rho \end{bmatrix} w(t) \\
y(t) &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1+0.1\rho \end{bmatrix} w(t) \\
\rho &\in [-1,1] \qquad \dot{\rho} \in [-1,1]
\end{aligned}
$$

- we choose $P(\rho) = P_0 + P_1\rho$ and we study γ w.r.t. the delay bound h_M

- We get the following figures



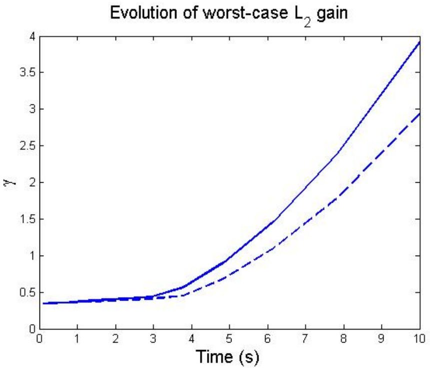


Example 2(3)

- Adding uncertainties

$$H_0 = H_1 = 0.1I, \qquad F_0 = F_1 = F_3 = F_4 = I$$
$$F_2 = F_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- We get



Conclusion et Future Works

Conclusion et Future Works

- Advantages (Stability/Performance Analysis) :
  - Simple and Fast
  - Interesting results but still conservative despite of the use of the Jensen's inequality.
- Use a more complex LKF, e.g.

$$
\begin{aligned}
V_2 &= \sum_{i=1}^{N} \int_{t-ih_n(t)}^{t-(i-1)h_n(t)} x(\theta)^T Q_i x(\theta) d\theta, \quad h_n(t) = h(t)/N \\
V_3 &= \bar{h} \sum_{i=1}^{N} \int_{t-i\bar{h}}^{t-(i-1)\bar{h}} \int_{t+\theta}^{t} \dot{x}(\eta)^T R_i \dot{x}(\eta) d\eta d\theta, \quad \bar{h} = h_M/N
\end{aligned}
$$

- Tackle the delay knowledge uncertainty