Positive systems analysis via integral linear constraints

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Positive systems analysis

- Quadratic forms are widely used for systems analysis: Lyapunov inequality, Kalman-Yakubovich-Popov Lemma, integral quadratic constraints etc.

- Analysis can be simplified if systems are known to be positive

- Lyapunov inequality:
  - $\exists P \succ 0$ such that $A^T P + PA \prec 0$
  - $\exists z > 0$ (element-wise) such that $Az < 0$

- Kalman-Yakubovich-Popov Lemma:
  - $\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* Q \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \quad \forall \omega \in [0, \infty]$
  - $\exists x, u, p \geq 0$ such that

$$Ax + Bu \leq 0 \quad \text{and} \quad Q \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p \leq 0$$

- The theory of integral linear constraints (ILCs)?

Outline

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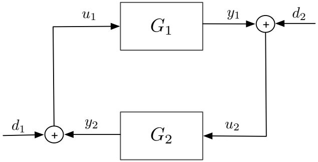
Positive systems

A system G is said to be positive if

$$u(t) \geq 0 \; \forall t \geq 0 \implies y(t) = (Gu)(t) \geq 0 \; \forall t \geq 0$$



Given a positive feedback interconnection of two positive systems G_1 and G_2, is the closed-loop map $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ always positive?

No!

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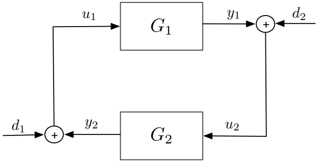

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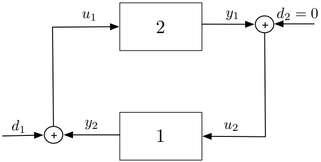
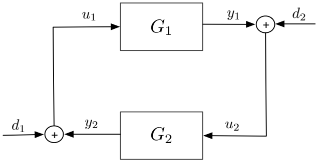
Positive systems

A simple counterexample:



$$d_1 \mapsto u_1 = \frac{1}{1-2} = -1$$

Feedback interconnections



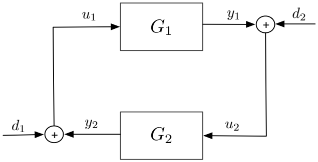
$$\hat{G}_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1$$
$$\hat{G}_2(s) = C_2(sI - A_2)^{-1}B_2 + D_2$$

- A_1 and A_2 are Metzler and $B_1 \geq 0$, $B_2 \geq 0$, $C_1 \geq 0$, $C_2 \geq 0$, $D_1 \geq 0$, and $D_2 \geq 0$ (element-wise) implies G_1 and G_2 are positive

Positivity of closed-loop map [Ebihara et. al. 2011]

If $\rho(D_1 D_2) < 1$, then $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ is positive

Feedback interconnections



Suppose (nonlinear) $G_i : \mathbf{L}_{1e} \to \mathbf{L}_{1e}$ are causal and positive, define

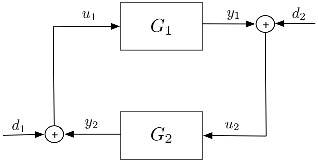
$$\alpha(G_i) := \sup_{T>0} \inf_{\Delta T>0} \sup_{\substack{x,y \in \mathbf{L}_{1e}; P_T x = P_T y \\ P_{T+\Delta T}(x-y) \neq 0}} \frac{\|P_{T+\Delta T}(G_i x - G_i y)\|_1}{\|P_{T+\Delta T}(x-y)\|_1}$$

Positivity of closed-loop map

If $\alpha(G_1)\alpha(G_2) < 1$, then $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ is positive

Outline

Robust stability of feedback systems



Integral quadratic constraints (IQCs) [Megretski & Rantzer 97]

Given bounded, causal $G_1 : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ and $G_2 : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$, suppose there exists linear $\Pi : \boldsymbol{L}_2 \to \boldsymbol{L}_2$ such that

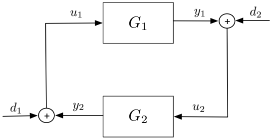
- $[\tau G_1, G_2]$ is well-posed for all $\tau \in [0, 1]$;
- $\int_0^\infty v(t)^T (\Pi v)(t)\, dt \geq 0 \quad \forall v \in \mathscr{G}(\tau G_1) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \boldsymbol{L}_2 : y = \tau G_1 u \right\}, \tau \in [0, 1]$;
- $\int_0^\infty w(t)^T (\Pi w)(t)\, dt \leq -\epsilon \int_0^\infty |w(t)|^2\, dt \quad \forall w \in \mathscr{G}'(G_2)$,

then $[G_1, G_2]$ is stable

Integral quadratic constraint (IQC) examples

Structure of G_1	Π	Condition
G_1 is passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\|G_1\| \leq 1$	$\begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix}$	$x(j\omega) \geq 0$
$G_1 \in [-1, 1]$	$\begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix}$	$X = X^* \geq 0,\ Y = -Y^*$
$G_1(t) \in [-1, 1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	$X = X^* \geq 0,\ Y = -Y^*$
$G_1(s) = e^{-\theta s} - 1$, for $\theta \in [0, \theta_0]$	$\begin{bmatrix} x(j\omega)\rho(\omega)^2 & 0 \\ 0 & -x(j\omega) \end{bmatrix}$	$\rho(\omega) = 2 \max_{\|\theta\| \leq \theta_0} \sin(\theta\omega/2)$

Robust stability of positive feedback systems



Integral linear constraints

Given bounded, causal, linear $G_1 : \mathbf{L}_{1e}^m \to \mathbf{L}_{1e}^p$ and $G_2 : \mathbf{L}_{1e}^p \to \mathbf{L}_{1e}^m$, suppose there exists $\Pi \in \mathbb{R}^{1\times m+p}$ such that

- $[\tau G_1, G_2]$ is well-posed and positive for all $\tau \in [0,1]$;
- $\int_0^\infty \Pi v(t)\,dt \geq 0 \quad \forall v \in \mathscr{G}_+(\tau G_1) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{L}_{1+} : y = \tau G_1 u \right\}, \tau \in [0,1]$;
- $\int_0^\infty \Pi w(t)\,dt \leq -\epsilon \int_0^\infty |w(t)|\,dt \quad \forall w \in \mathscr{G}_+'(G_2)$,

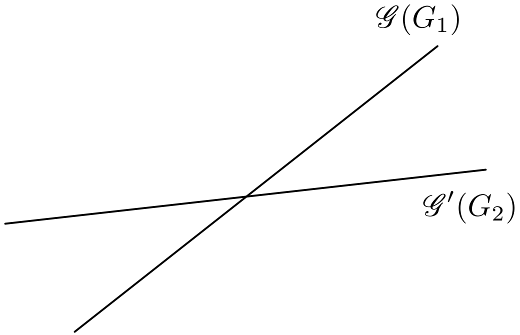
then $[G_1, G_2]$ is stable

When G_1 and G_2 are LTI, conditions can be stated as

- $\Pi \begin{bmatrix} I \\ \tau \hat{G}_1(0) \end{bmatrix} \geq 0 \quad$ and $\quad \Pi \begin{bmatrix} \hat{G}_2(0) \\ I \end{bmatrix} < 0$

Outline

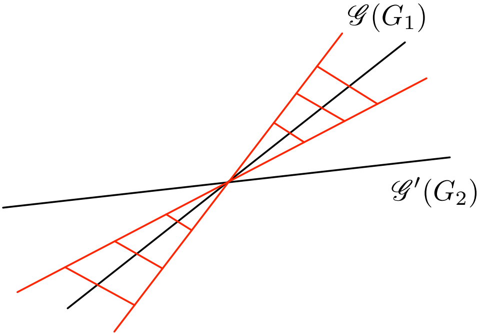
Geometric interpretation of integral quadratic constrains



Feedback stability

- $\mathscr{G}(G_1) + \mathscr{G}'(G_2) = \boldsymbol{L}_2$;
- $\mathscr{G}(G_1) \cap \mathscr{G}'(G_2) = \{0\}$

Geometric interpretation of integral quadratic constraints



Integral quadratic constraints (IQCs)

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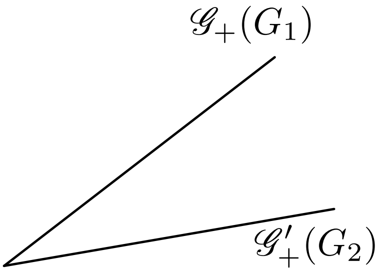

Feedback stability

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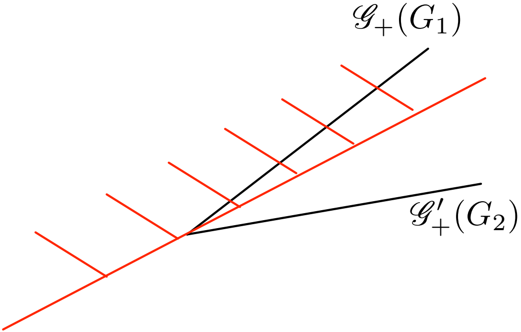

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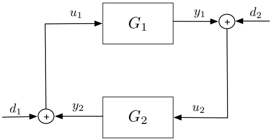
LTI systems



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- A_1 and A_2 are Metzler, Hurwitz and $B_1 \geq 0$, $B_2 \geq 0$, $C_1 \geq 0$, $C_2 \geq 0$, $D_1 \geq 0$, and $D_2 \geq 0$

Robust stability [Ebihara et. al. 2011] [Tanaka et. al. 2013]

If $\rho(\hat{G}_1(0)\hat{G}_2(0)) < 1$, then $[G_1, G_2]$ is stable

Can be recovered with integral linear constraint theorem with

$$\Pi := z^T \begin{bmatrix} \hat{G}_1(0) & -I \end{bmatrix},$$

where $z^T(\hat{G}_1(0)\hat{G}_2(0) - I) < 0$

Conclusions:

- Sufficient condition for positivity to be preserved under feedback
- Developed integral linear constraints theory for analysis of feedback interconnections with positive closed-loop mappings
- Many extensions possible:
  - Positive coprime factorisations
  - Integral linear constraints with time-varying multipliers
  - LMI conditions for verifying integral linear constraints
  - Stabilisation of open-loop unstable dynamics?