Design of $\mathcal{H}_\infty$ Bounded Non-Fragile Controllers for Discrete-Time Systems

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Plan

- Introduction
- Main Results
- Examples

Introduction

Previous Works

- Resilience of controllers [Keel et al. '97]
  - Continuous-Time systems
    - Ricatti [Haddad, 97], [Yang et al, 01]
    - LMI [Jadbabaie et al. 97], [Peaucelle et al. 04]
  - Discrete-Time Systems ?
- Bounded controller design (NP-hard, [Blondel et al. 97])
  - Sporadic results, e.g. [Peaucelle et al. 08] in continuous-time
  - Discrete-time ?

Goals

- Resilient Controllers (SF) Synthesis for DT systems
- Bounded Controllers Synthesis
- Efficient characterization of solutions (LMIs)
- Include performance optimization

Systems

- Discrete-time linear systems

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix}$$

  state x, control input u, exogenous input w, controlled output z.
- Matrices supposed known
- Can be extended easily to the uncertain case

Control Laws

- Find a control law of the form

$$u(k) = Kx(k)$$

  such that it
  - stabilizes the system
  - minimizes a performance criterium, e.g. $\mathcal{H}_\infty$.
- Moreover, the controller must also satisfy
  - A resilience (non-fragility) property
  - A boundedness condition for the coefficients

Non-fragility property

- Self-robustness property of the controller
- Error on the controller implementation gain maintain closed-loop stability
- Model of the implementation error
- Two type of errors :
  - Additive error (rounding, uniform discrete valued space)

$$K_i = K_c + \delta K$$

  - Additive and multiplicative error (rounding+nonuniform discrete valued space)

$$K_i = K_c + \theta K_c + \Gamma$$

K_i implemented controller, K_c computed one, $\delta K, \theta, \Gamma$ error terms

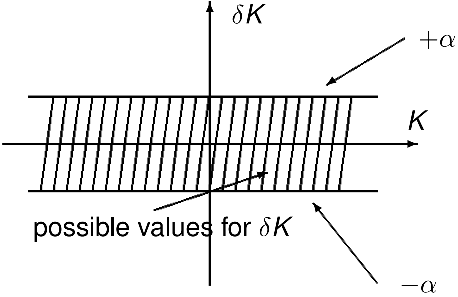
Additive Error

- Form of implemented gain

$$K_i = K_c + \delta K \qquad \delta K = U \Delta V$$

$$\Delta \text{ diagonal}, \ ||\Delta||_2 \leq \alpha$$

- Coefficients of δK inside $[-\alpha, \alpha]$



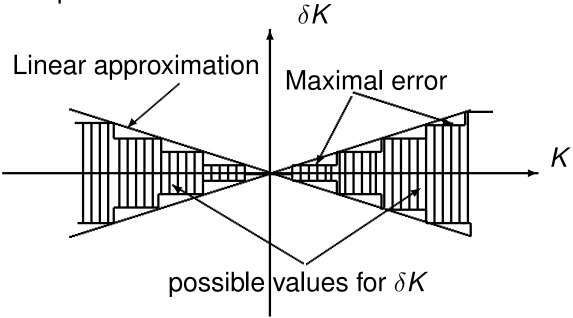
Additive-Multiplicative Error

- Form of implemented gain

$$K_i = (1+\theta)K_c + \Gamma \qquad \Gamma = U\tilde{\Delta}V$$

$$\theta \in [-\mu, \mu] \quad ||\tilde{\Delta}||_2 \leq \tilde{\alpha}$$

- Illustration of possible error behavior :



δK total implementation error

Bounded coefficients

- Form of implemented gain (with additive error)

$$K_i = \underbrace{M_1(K_0 + K_c)M_2}_{\text{previous } K_c} + \delta K$$

M_1, M_2 scaling terms, K_0 shifting term

- K_0 allows for looking for a controller centered around 0 such that

$$||K_c + M_1^{-1}\delta K M_2^{-1}||_2 \leq \beta\sqrt{mn}$$

m, n dimensions of input and state resp.

Main Results

Resilient state-feedback (additive)

Theorem

There exists a quadratically stabilizing resilient state-feedback if there exist a matrix $X = X^T \succ 0$, a diagonal matrix $Q \succ 0$ and a scalar $\gamma > 0$ such that the following LMI

$$\begin{bmatrix} -X & 0 & XV^T & \mathcal{M}_{14} & \mathcal{M}_{15} \\ \star & -\gamma I & 0 & F^T & E^T \\ \star & \star & -Q & 0 & 0 \\ \star & \star & \star & -\gamma I + \alpha^2 DUQU^T D^T & \alpha^2 DUQU^T B^T \\ \star & \star & \star & \star & -X + \alpha^2 BUQU^T B^T \end{bmatrix} \prec 0$$

holds where

$$\begin{array}{rcl} \mathcal{M}_{15} & = & [AX + BM_1 K_0 M_2 X + BM_1 Y]^T \\ \mathcal{M}_{14} & = & [CX + DM_1 K_0 M_2 X + DM_1 Y]^T \end{array}$$

In such a case, we have $K_c = Y(M_2 X)^{-1}$ and the closed-loop system satisfies $||z||_{\ell_2} \leq \gamma ||w||_{\ell_2}$.

Sketch of the proof

- Write the closed-loop system
- Substitute into the BRL
- Rewrite the BRL into the form

$$\Psi + \mathcal{U}^T \Delta \mathcal{V} + \mathcal{V}^T \Delta^T \mathcal{U} \prec 0$$

- Apply the Petersen's lemma (or Scaled-bounded real lemma), congruence transformations, Schur complement and change of variables (standard)

Adding constraints on the controller coefficients (1)

- Idea : Add a condition to the previous design $\rightarrow$ add-on
- Nonlinear constraint on the controller (proved NP-hard, nonconvex) $\rightarrow$ no exact LMI formulation
- Relaxation necessary (Cone complementary algorithm or iterative LMI algorithm)

Adding constraints on the controller coefficients (2)

- Iterative LMI based result (no additional optimization cost)

Theorem

Find N, Y and $X \succ 0$ of appropriate dimension such that

$$\begin{bmatrix} \Pi_{11} & Y & 0 & 0 \\ \star & N^T M_2 X + X M_2^T N & X M_2^{-T} V^T & N^T \\ \star & \star & -H & 0 \\ \star & \star & \star & -I \end{bmatrix} \preceq 0$$

$$\Pi_{11} = -s^2 mn\beta^2 I + \alpha^2 M_1^{-1} UHU^T M_1^{-T}$$

This will result in a gain K_c satisfying $||K_c + M_1^{-1}\delta K M_2^{-1}||_2 \leq s\sqrt{mn}\beta$.

- Iteration between X and the slack-variable N
- Can be proved using the projection lemma.

Example

Example (1)

- Let us consider the unstable system

$$\begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k) + Ew(k) \\ z(k) & = & Cx(k) + Du(k) + Fw(k) \end{array}$$

with matrices $F = 0$

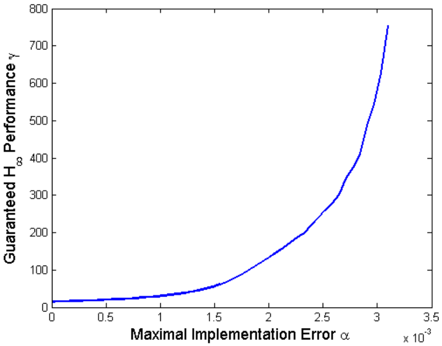
$$A = \begin{bmatrix} 9.3547 & 0.5789 & 1.3889 & 2.7219 \\ 9.1690 & 3.5287 & 2.0277 & 1.9881 \\ 4.1027 & 8.1317 & 1.9872 & 0.1527 \\ 8.9365 & 0.0986 & 6.0379 & 7.4679 \end{bmatrix}$$

$$C = 0.1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \quad E = 0.1 \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example (2)

- With no implementation error we get $\gamma^* = 14.78$
- System stabilizable for all $\alpha < 0.0032$ (need quite large precision)
- For a precision of $\alpha = 0.0020$, we find $\gamma_a = 132.9090$ (worst case)
- After rounding and verification, we get $\gamma_r = 88.4425$



- Optimal Controller

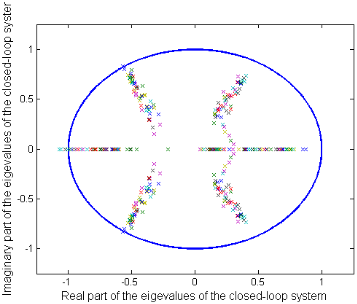
$$K^* = \begin{bmatrix} -18.6097 & -3.5441 & -5.8235 & -7.7459 \\ -2.3537 & -3.0544 & -0.8132 & -0.0420 \end{bmatrix}$$

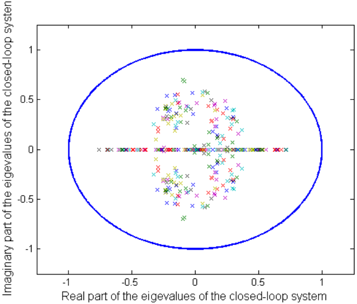
- Resilient Controller

$$K_a = \begin{bmatrix} -53.4820 & -16.3820 & -20.9800 & -24.6260 \\ 42.0900 & 13.3060 & 18.5000 & 21.4660 \end{bmatrix}$$

Example (3)

- Location of eigenvalues of the closed-loop system for random implementation error lower than 0.0025.





Left : optimal controller, Right : Memory resilient controller

- unstable behavior on the left

Conclusion and Future Works

Conclusion and Future Works

- Characterization of Resilient SF Controllers
- Two types of error
- LMI form (optimization)
- Additional nonlinear constraint for the boundedness of controllers (relaxation)

- Characterize more general class of errors
- Dynamic Output Feedback case
- Other formulations for boundedness of controllers (more relevant in continuous time)

Thank you for your attention