Robust stability of impulsive systems: A functional-based approach

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Outline

Linear impulsive systems

Consider a linear hybrid system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \ t \in \mathbb{R}_+ \backslash \mathbb{I}, \\ x(t^+) &= Jx(t^-), \ t \in \mathbb{I}, \end{aligned} \quad (1)$$

- $x \in \mathbb{R}^n$ is the state;
- A and J are matrices of appropriate dimensions
- the set $\mathbb{I}$ represents a strictly increasing sequence of instants $\{t_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to +\infty} t_k = +\infty$.

This class of systems occurs in several fields like epidemiology [?, ?], sampled-data and networked control systems [?], etc.

Periodic impulse instants

Assume that the impulses are periodic i.e. $t_{k+1} - t_k = T = Cst$. Then the stability analysis can be performed as follows

$$\mathbf{C1} : \rho(Je^{AT}) < 1 \text{ OR } \rho(Je^{AT}) < 1$$

An alternative analysis uses Lyapunov Theorem and relies on the existence of a positive definite symmetric matrix P such that the LMI condition

$$\mathbf{C2} : (Je^{AT})^{\intercal} P J e^{AT} - P < 0$$

holds.

Note that these conditions are necessary and sufficient.

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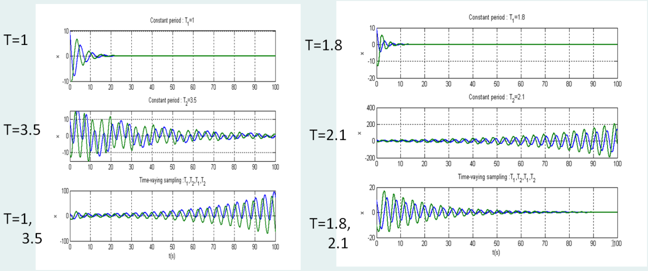
Question:

What happens in the aperiodic case?

Aperiodic impulse instants

The condition **C1** is not sufficient in this case. Consider the following example (inspired from the time-delay and sampled-data systems[**?**, **?**])

$$A = \begin{bmatrix} A_0 & B_0 \\ 0 & 0 \end{bmatrix},\ J = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix},\ A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$



Proposed solution

Stability can still be analyzed using Lyapunov theory.

Discrete-time Lyapunov Theorem

If, for some given positive scalar T_{min}, T_{max}, there exists a symmetric positive definite matrix P such that

$$(e^{A\theta})^{\intercal} J^{T} P J e^{A\theta} - P < 0$$

for all $\theta \in [T_{min}, T_{max}]$. Then, the impulse system is asymptotically stable for all impulse instants sequence $\{t_k\}_{k\in\mathbb{N}}$ verifying $t_{k+1} - t_k \in [T_{min}, T_{max}]$

Existing solutions

- Gridding+robust analysis : [?],...
- Approximation of exponential terms: [?, ?, ?],...

Robust stability

The above methods work well in the case of constant and known matrices A and J. They are, however, difficult to extend to uncertain systems due to the exponential terms in the Lyapunov conditions.

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$\Rightarrow$ Need for a new methodology

Define the set $\mathbb{K}$ representing the set of continuous functions of the form

$$\chi_k : [0, T_k) \to \mathbb{R}^n,$$

where $T_k = t_{k+1} - t_k < \infty$, $k \in \mathbb{N}$.
Consider now the impulsive system in a lifted representation as suggested in [?] for sampled-data systems. The new (lifted) state-space is hence infinite-dimensional and verifies

$$\begin{array}{rcl} \chi_k(\tau) & := & x(t_k^+ + \tau), \\ \chi_k(\tau) & = & e^{A\tau}\chi_k(0), \\ \chi_{k+1}(0) & = & J\chi_k(T_k^-) = Jx(t_{k+1}^-). \end{array}$$

The function χ_k represents the trajectory of the system between two successive impulse instants.

Theorem

Let $0 < T_{min} \leq T_{max} < +\infty$ *and* $V : \mathbb{R}^n \to \mathbb{R}^+$ *be a quadratic function. Then the two following statements are equivalent.*

(i) *The sequence* $\{V(x(t_k^-))\}_{k\in\mathbb{N}}$ *is decreasing;*

(ii) *There exists a functional* $\mathcal{V} : [0, T_{max}] \times \mathbb{K} \to \mathbb{R}$ *satisfying*

$$\mathcal{V}(T, z) = \mathcal{V}(0, z), \tag{2}$$

for all $z \in \mathbb{K}$ *and for all* $T \in [T_{min}, T_{max}]$ *and such that*

$$\begin{array}{rcl} \dot{\mathcal{W}}_k(\tau, \chi_k) & := & \Lambda_k + \frac{\mathrm{d}}{\mathrm{d}\tau}[T_k V(\chi_k(\tau)) + \mathcal{V}(\tau, \chi_k)] < 0, \\ \Lambda_k & := & V(\chi_k(0)) - V(\chi_{k-1}(T_{k-1}^-)) \end{array}$$

holds for all $\tau \in [0, T_k)$, $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.
Moreover, if one of the previous items is satisfied, the impulsive system (1) *is asymptotically stable.*

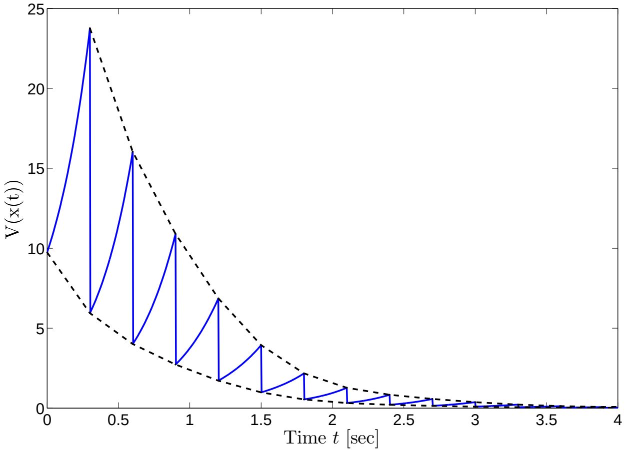

Figure : Continuous-time Lyapunov function V (blue) for system (1) and the discrete-time envelopes (black); $\mathcal{W}$ coincides with the monotonically decreasing lower envelope.

ii⇒i

There exists a functional $\mathcal{V} : [0, T_{max}] \times \mathbb{K} \to \mathbb{R}$ satisfying $\mathcal{V}(T, z) = \mathcal{V}(0, z)$, for all $z \in \mathbb{K}$ and for all $T \in [T_{min}, T_{max}]$ and such that

$$\dot{\mathcal{W}}_k(\tau, \chi_k) < 0,$$

holds for all $\tau \in [0, T_k)$, $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$

Integrate $\dot{\mathcal{W}}_k$ over the interval $[0, \ T_k]$ leads to

$$\begin{aligned} \int_0^{T_k^-} \dot{\mathcal{W}}_k(\tau, \chi_k) \mathrm{d}\tau &= T_k \left[\Lambda_k + V(x(t_{k+1}^-)) - V(x(t_k^+)) \right. \\ &\quad \left. + \mathcal{V}(T, \chi_k) - \mathcal{V}(0, \chi_k)\right] \\ &= T_k \left[V(x(t_{k+1}^-)) - V(x(t_k^-))\right]. \end{aligned}$$

where $\Lambda_k = V(x(t_k^+)) - V(x(t_k^-))$.
$\Rightarrow$ Then the sequence $\{V(x(t_k^-))\}_{k \in \mathbb{N}}$ is decreasing over k since $\dot{\mathcal{W}}_k$ is negative over $[0, T_k)$.

i⇒ii

$\{V(x(t_k^-))\}_{k\in\mathbb{N}}$ is strictly decreasing.

In other words, **i** means that

$$\mathcal{I}(P, A, J, T_k) := (e^{AT_k})^T J^T P J e^{AT_k} - P \prec 0$$

for all $T_k \in [T_{min}, T_{max}]$.
Then consider the functional

$$\mathcal{V}(\tau, \chi_k(\tau)) = -T_k V(\chi_k(\tau)) + \tau(V(\chi_k(T_k)) - V(\chi_k(0^+)))$$

and from the definition of the functional $\dot{\mathcal{W}}_k$ we get

$$\dot{\mathcal{W}}_k(\tau, \chi_k) \; = \; x^T(t_k)\left[(e^{AT_k})^T J^T P J e^{AT_k} - P\right] x(t_k) < 0.$$

Asymptotic stability : the trajectories of the system do not diverge in finite time (before T_{max}).

Interests of the proposed result

- Discrete-time stability analysis based on the continuous-time formulation of the system.
- No exponential matrix e^{AT};
- The functional $\mathcal{V}$ **is not needed to be positive definite**. It only has to satisfy the boundary conditions (unlike usual functionals for impulsive systems).
- Extension to robust stability of impulsive systems is straightforward due to the convexity of the conditions.

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Objectives

Define a class of functional $\mathcal{V}$ that satisfies **ii** in order to ensure **i**.

An example of functionals

Choosing $V(x) = x^T P x$ and a functional $\mathcal{V}$, inspired from [?],[?].

$$\begin{aligned}\mathcal{V}(\tau, \chi_k) &= (T-\tau)\zeta_k(\tau)^T \left[Q\zeta_k(\tau) + 2R\chi_{k-1}(T_{k-1}^-)\right] \\ &\quad +(T-\tau)\int_0^\tau \dot{\chi}_k(s)^T Z \dot{\chi}_k(s) ds \\ &\quad +\tau(T-\tau)\chi_{k-1}^T(T_{k-1}^-) U \chi_{k-1}(T_{k-1}^-),\end{aligned} \tag{3}$$

where $\zeta_k(\tau) = \chi_k(\tau) - \chi_k(0) = \chi_k(\tau) - J\chi_{k-1}(T_{k-1}^-)$,
$P = P^T \succ 0,\ Q = Q^T,\ Z = Z^T \succ 0,\ U = U^T,\ R.$

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This functional satisfies the boundary condition

$$\mathcal{V}(T,\chi_k) = \mathcal{V}(0,\chi_k) = 0$$

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Then classical differentiation and computations leads to the following conditions.

Theorem : Periodic impulses instants and known A and J

The impulsive system (1) is AS if there exist $P, Z \in \mathbb{S}^n_{++}$, $Q, U \in \mathbb{S}^n$, $R \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times 2n}$ such that

$$\begin{array}{rcll} \Psi(A,J,T) & := & F_0 + T(F_2+F_3) & \prec 0, \\ \Phi(A,J,T) & := & \begin{bmatrix} F_0 - TF_3 & TN^T \\ \star & -TZ \end{bmatrix} & \prec 0, \end{array} \quad (4)$$

hold with $M_x = \begin{bmatrix} I & 0 \end{bmatrix}$, $M_- = \begin{bmatrix} 0 & I \end{bmatrix}$, $M_\zeta = \begin{bmatrix} I & -J \end{bmatrix}$, $F_3 = M_-^T U M_-$ and

$$\begin{array}{l} F_0 = T\mathrm{He}\{M_x^T PAM_x - N^T M_\zeta - M_\zeta^T RM_-\} - M_\zeta^T QM_\zeta \\ \qquad +M_-^T(J^TPJ - P)M_-, \\ F_2 = \mathrm{He}[M_x^T A^T QM_\zeta + M_x^T A^T RM_-] + M_x^T A^T ZAM_x. \end{array} \quad (5)$$

The periodic impulsive system is asymptotically stable and the LMI

$$\mathcal{I}(P,A,J,T) \prec 0$$

Assume now that the matrices A and J belong to the convex polytopes:

$$A \in \mathcal{A} := \mathrm{co}\{A_1, \ldots, A_N\}, \quad J \in \mathcal{J} := \mathrm{co}\{J_1, \ldots, J_M\} \tag{6}$$

and $T \in [T_{min}, T_{max}]$. Then, we have the following result:

Theorem : Aperiodic impulses instants and uncertain A and J

The impulsive system (1) is asymptotically stable if there exist $P, Z \in \mathbb{S}^n_{++}$, $Q, U \in \mathbb{S}^n$, $R \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^{n \times 2n}$ such that the LMIs

$$\Psi(A_i, J_j, T) \prec 0, \quad \Phi(A_i, J_j, T) \prec 0, \qquad \forall i \in \{N\}, \ \forall j \in \{M\} \tag{7}$$

hold for $T \in \{T_{min}, T_{max}\}$.
Moreover, $V(x) = x^T P x$ is a discrete-time Lyapunov function for system (1), that is the LMI $\mathcal{I}(P, A, J, T) \prec 0$ holds.

A anti-Hurwitz and J Schur

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \qquad J = 0.5 I_2. \tag{8}$$

A Hurwitz and J anti-Schur

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \qquad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}. \tag{9}$$

A not (anti-) Hurwitz or and J (anti-)Schur

$$A = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \qquad J = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{10}$$

Periodic Case	Ex.1	Ex.2	Ex.3
Th. upper bound	$(0\ 0.4620]$	$[1.140\ +\infty)$	$[0.1824\ 0.5776]$
[?]	$(0\ 0.4471]$	$[1.232\ +\infty)$	$[0.1824\ 0.5760]$
Theorem 2	$(0\ 0.4519]$	$[1.174\ +\infty)$	$[0.1824\ 0.5764]$

Table : Admissible dwell-time in the periodic case.

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[?]	$(0\ 0.4471]$	$[1.232\ \ +\infty)$	$[0.1907\ 0.5063]$
Theorem 2	$[10^{-6}\ 0.4483]$	$[1.232\ 10^{6}]$	$[0.1824\ 0.5741]$

Table : Admissible dwell-time in the aperiodic case.

A anti-Hurwitz and J Schur

Let us consider an uncertain version of the system treated in Example 1. Now the matrix $A \in \mathcal{A}$ where

$$\mathcal{A} := \mathrm{co}\left\{ \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 6 \end{bmatrix} \right\} \quad J = 0.5 I_2..$$

Periodic Case	Ex.1
"Th. upper bound" / Periodic	(0 0.1155]
Theorem 3 / Periodic	(0 0.1149]
Theorem 3 / Aperiodic	$(10^{-6}$ 0.1148]

Table : Allowable dwell time for the uncertain impulsive system for the periodic and aperiodic cases.

An additional example is provided in the paper to consider the case of uncertain matrix J.

What has been proposed

- A new framework for the analysis of impulsive systems
- Robust stability results (dwell-time) for impulsive systems

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What has to be done

- Nonlinear systems