Dynamic equations on time-scale: application to stability analysis and stabilization of aperiodic sampled-data systems

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Outline

- Introduction
- Problem statement and Preliminaries
- Stability analysis
- Stabilization
- Conclusion and Future Works

Aperiodic sampled-data systems

- Discrete-time systems with varying sampling period
- Several frameworks
  - Time-delay systems [Yu et al.], [Fridman et al.]
  - Impulsive systems [Naghshtabrizi et al.], [Seuret]
  - Sampled-data systems [Mirkin]
  - Robust techniques [Fujioka], [Hetel et al.], [Oishi et al.], [Ariba et al.]
  - Functional-based approaches [Seuret]

Problem statement

System and Problem definition

- Continuous-time LTI system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & x_0 \end{array} \qquad (1)$$

with state x and control input u.

- Sampled-data control law

$$u(t) = Kx(t_k),\ t \in [t_k, t_{k+1}) \qquad (2)$$

where $T_k := t_{k+1} - t_k \leq T$, $k \in \mathbb{N}$.

- Stability analysis problem: given K, find the set $\mathcal{T}$ of admissible $T > 0$ for which stability still holds.
- Find controller gain K maximizing the maximal sampling period.

Systems over time-scales [Bohner]

- Unification/generalization of continuous-time and discrete-time systems
- Examples: $\mathbb{T} = \mathbb{R}_+$, $\mathbb{T} = \mathbb{Z}_+$, $\mathbb{T} = \{0\} \cup \{1/k\}_{k\in\mathbb{N}}$, $\mathbb{T} = \bigcup_{k\in\mathbb{N}} [t_{2k}, t_{2k+1}]$
- Forward jump operator: $\sigma(t) = \{\inf s \in \mathbb{T} : t < s\}$
- Graininess: $\mu(t) = \sigma(t) - t$ (distance)
- Dynamical system on time-scale:

$$\begin{array}{rcl} x^{\Delta}(t) & = & Ax(t) + Bu(t) \\ x(0) & = & x_0 \end{array} \tag{3}$$

- Δ operator [Goodwin]:

$$f^{\Delta}(t) := \begin{cases} \lim\limits_{s\to t, s\in\mathbb{T}} \dfrac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0 \\ \dfrac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) > 0. \end{cases}$$

Stability analysis via Lyapunov functions

- Linear systems $\rightarrow V(x) = x^T P x$
- Stability condition

$$A^T P + PA + \mu A^T PA \prec 0, \quad P = P^T \succ 0 \tag{4}$$

- μ fixed: equivalent to a DT criterion
- $\mu \rightarrow 0$: equivalent to a CT criterion
- Spectrum condition: $\lambda(A) \subset \mathbb{D}(-1/\mu, 1/\mu)$
  - $\mu \rightarrow 0$: $\mathbb{D}(-1/\mu, 1/\mu) \rightarrow \mathbb{C}_-$
  - $\mu = \mu_0 \neq 0$: $\mathbb{D}(1/\mu_0, 1/\mu_0)$ analogous to the unit disc.

Representation of sampled-data systems

- DT system:

$$x(t) = \left[e^{A(t-t_k)} + \int_{t_k}^{t} e^{A(t-s)} ds BK(t_k)\right] x(t_k) \tag{5}$$

- System on TS:

$$z^{\Delta}(t_k) = A_{\Delta}(\mu(t_k))z(t_k) \tag{6}$$

where the new state z coincide with x at sampling instants and

$$\begin{aligned} A_{\Delta}(\mu(t_k)) &= \mu(t_k)^{-1}\left(e^{A\mu(t_k)} + \Phi(\mu(t_k))BK(t_k) - I\right) \\ \Phi(\mu(t_k)) &= \int_{0}^{t_k} e^{A(\mu(t_k)-s)} ds \end{aligned} \tag{7}$$

Stability analysis

A general stability result

Theorem

The dynamical system $z^{\Delta}(t_k) = A_{\Delta}(t_k)z(t_k)$, $z(t_0) = z_0$, $(t_0, t_k) \in \mathbb{T}^2$, $t_k \geq t_0$ *is robustly exponentially stable for* $\mu(t_k) \in \boldsymbol{\mu}$ *if the following statements hold:*

1. $A(t_k)$ *is rd-continuous and regressive, i.e.* $\det(I + \mu(t_k)A(t_k)) \neq 0$ *for all* $t_k \in \mathbb{T}$ *and* $\mu(t_k) \in \boldsymbol{\mu}$.
2. *There exist* $P : \mathbb{T} \to \mathbb{S}^n_{++}$ *verifying* $\theta_1 I \preceq P(t_k) \preceq \theta_2 I$ *for some* $0 < \theta_1 < \theta_2 < +\infty$ *and* $\beta \in (0, 1/\sup\{\boldsymbol{\mu}\})$ *such that*

$$\mathcal{M}_{\mu(t_k)}(P(\sigma(t_k)), A_{\Delta}(t_k), P^{\Delta}(t_k) + \beta P(t_k)) \preceq 0 \tag{8}$$

holds for all $\mu(t_k) \in \boldsymbol{\mu}$ *and all* $t_k \in \mathbb{T}$.

Graininess dependent Lyapunov function

Theorem

The dynamical system $z^{\Delta}(t_k) = A_{\Delta}(\mu(t_k))z(t_k)$, $z(t_0) = z_0$, $(t_0, t_k) \in \mathbb{T}^2$, $t_k \geq t_0$ is robustly exponentially stable for $\mu(t_k) \in \boldsymbol{\mu}$, $\inf\{\boldsymbol{\mu}\} > 0$, if the following statements hold:

1. *$A(\mu(t_k))$ is rd-continuous and regressive, i.e. $\det[I + \mu A(\mu)] \neq 0$ for all $\mu \in \boldsymbol{\mu}$.*
2. *There exist a bounded matrix function $P : \boldsymbol{\mu} \to \mathbb{S}^n_{++}$ such that*

$$\mathcal{M}_{\mu}(P(\mu_n), A_{\Delta}(\mu_c), S(\mu_c, \mu_n)) \prec 0 \tag{9}$$

holds for all $(\mu_n, \mu_c) \in \boldsymbol{\mu}^2$ and where $S(\mu_c, \mu_n)) = \mu_c^{-1}(P(\mu_n) - P(\mu_c))$.

Example 1

- Scalar system ([Mirkin], [Fridman], [Fujioka])

$$\dot{x}(t) = -2x(t) + x(t_k). \tag{10}$$

- TS formalism

$$A_{\Delta}(\mu) = \frac{3}{2\mu}\left(e^{-2\mu} - 1\right). \tag{11}$$

- Lyapunov condition

$$\frac{3}{\mu}\left(e^{-2\mu} - 1\right)\left(1 + \frac{3}{4}\left(e^{-2\mu} - 1\right)\right) < 0$$

- True for $\mu > 0$
- When $\mu \to 0$, the LHS tends to -6.

Example 2

Let us consider the system

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} \qquad (12)$$

Ref.	Maximal varying sampling period
[Fridman,04]	0.8696
[Yue,05]	0.8871
[Ariba,07	1.009
[Naghshtabrizi,08]	1.1137
[Mirkin,07]	1.3659
[Seuret,09b]	1.6894
[Oishi,09]	1.7294
Proposed result	1.72941
Theoretical	1.7294194 (constant case)

Example 3

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

- Pathological sampling periods: $\{2.2228, 4.4457, 6.6685, 8.8913, 11.1142, \dots\}$,
- Constant sampling period
  - $P(\mu) = P_0 \rightarrow \boldsymbol{\mu} = [0.5004, 1.9203]$
  - $P(\mu) = P_0 + P_1\mu \rightarrow \boldsymbol{\mu} = [0.2013, 2.0204]$
  - Other disjoint intervals

$$\begin{aligned} \boldsymbol{\mu} \quad \in \quad & \{[2.4706, 3.6963], [5.4307, 6.3447], [7.0277, 7.7249], \\ & [10.3916, 10.7559], [11.4973, 11.7179]\}. \end{aligned}$$

- Aperiodic case:

$$\begin{aligned} \boldsymbol{\mu} \quad \in \quad & \{[0.2187, 1.0031], [0.500, 1.9256], [2.47, 3.6], \\ & [2.77, 3.6963], [5.4584, 5.8004], [5.8172, 6.3447], \\ & [7.0339, 7.5009], [7.5000, 7.7070]\}. \end{aligned} \tag{13}$$

Stabilization

Robust state-feedback design

Theorem

There exists a quadratically stabilizing switching sampling-period-dependent state-feedback control law if there exist $X \in \mathbb{S}^n_{++}$ *and a bounded continuous matrix function* $U : \boldsymbol{\mu} \to \mathbb{R}^{n \times m}$ *such that the LMI*

$$\begin{bmatrix} \Xi_{11}(\mu) & \Xi_{12}(\mu) \\ \star & -\mu^{-1}X \end{bmatrix} \prec 0 \tag{14}$$

holds for all $\mu \in \boldsymbol{\mu}$ *with*

$$\begin{array}{rcl} \Xi_{12}(\mu) & = & \mu^{-1}[A_e(\mu)X + \Phi(\mu)BU(\mu)]^T \\ \Xi_{11}(\mu) & = & \Xi_{12}(\mu) + \Xi_{12}(\mu)^T \\ A_e(\mu) & = & \exp(A\mu) - I \end{array} \tag{15}$$

In such a case, the controller matrix is given by $K(\mu) = U(\mu)X^{-1}$.

Sampling-period dependent controller

Theorem

There exists a robustly stabilizing sampling-period-dependent state-feedback control law if there exist a matrix $Z \in \mathbb{R}^{n\times n}$*, bounded continuous matrix functions* $P : \boldsymbol{\mu} \to \mathbb{S}^n_{++}$, $U : \boldsymbol{\mu} \to \mathbb{R}^{n\times m}$ *and a sufficiently large positive scalar function* $\epsilon : \boldsymbol{\mu}^2 \to \mathbb{R}_{++}$ *such that the matrix inequality*

$$\begin{bmatrix} \Xi_{11}(\mu_c,\mu_n) & \Xi_{12}(\mu_c,\mu_n) & Z \\ \star & \Xi_{22}(\mu_c,\mu_n) & 0 \\ \star & \star & \Xi_{33}(\mu_c,\mu_n) \end{bmatrix} \prec 0 \tag{16}$$

holds for all $\mu \in \boldsymbol{\mu}$, $\inf\{\boldsymbol{\mu}\} > 0$, *with* $S(\mu_c,\mu_n) = \mu_c^{-1}(P(\mu_n) - P(\mu_c))$ *and*

$$\begin{array}{rcl} \Xi_{11}(\mu_c,\mu_n) & = & -Z - Z^T + \mu_c P(\mu_n) \\ \Xi_{12}(\mu_c,\mu_n) & = & \mu_c^{-1}\left[A_e(\mu_c)X + \Phi(\mu_c)BU(\mu_c)\right] + P(\mu_n) \\ \Xi_{22}(\mu_c,\mu_n) & = & -\epsilon(\mu_c,\mu_n)P(\mu_n) + S(\mu_c,\mu_n) \\ \Xi_{33}(\mu_c,\mu_n) & = & -P(\mu_n)/\epsilon(\mu_c,\mu_n) \\ A_e(\mu_c) & = & \exp(A\mu_c) - I \end{array} \tag{17}$$

In such a case, the controller matrix is given by $K(\mu_c) = U(\mu_c)Z^{-1}$.

Examples

- System 1:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{18}$$

- System 2:

$$A = \begin{bmatrix} 7 & 4 \\ 5 & 11 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{19}$$

Examples

degree of $K(\mu)$	μ^+ for System 1	μ^+ for System 2
0	1.8938	0.1566
1	2.2072	0.3299
2	2.2121	0.5020
3	2.2170	0.6717
4	2.2182	0.8353
5	2.2206	0.9817
6	2.2206	1.0196

Conclusion

Conclusion

- Time-scale approach for stability analysis of sampled-data systems
- Stability analysis via sampling-period dependent Lyapunov Functions
- Stabilization via sampling-period dependent controllers
- Extension to dynamic output feedback possible
- Future works will be devoted to the theory of systems on time-scales

Thank you for your attention !