Stability Criteria for Asynchronous Sampled-data Systems - A Fragmentation Approach

C. Briat and A. Seuret

KTH, Stockholm, Sweden
GIPSA-Lab, Grenoble, France

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Outline

- Introduction
- Problem statement and Preliminaries
- Stability analysis
- Conclusion and Future Works
Aperiodic sampled-data systems

- Discrete-time systems with varying sampling period
- Several frameworks
  - Time-delay systems [Yu et al.], [Fridman et al.]
  - Impulsive systems [Naghshtabrizi et al.], [Seuret]
  - Sampled-data systems [Mirkin]
  - Robust techniques [Fujioka], [Oishi et al.], [Ariba et al.]
  - Functional-based approaches [Seuret]
Problem statement
Continuous-time LTI system

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
x(0) = x_0
\] (1)

with state \( x \) and control input \( u \).

Sampled-data control law

\[
u(t) = Kx(t_k), \ t \in [t_k, t_{k+1})
\] (2)

where \( T_k := t_{k+1} - t_k \in \mathcal{T} := [T_{min}, T_{max}], \ k \in \mathbb{N} \).

Stability analysis problem: given \( K \), find the set \( \mathcal{T} \) for which for all \( T_k \in \mathcal{T} \) stability holds.
It is straightforward to show that the system is asymptotically stable if there exists $P = P^T \succ 0$ such that the LMI

$$\Phi(T)^T P \Phi(T) - P \prec 0$$

for all $T \in \mathcal{T}$ and where

$$\Phi(T) = e^{AT} + \int_0^T e^{A(T-s)} ds BK$$

(3)

- LMI difficult to check (although possible [Fujioka])
- Difficult to extend to uncertain systems or nonlinear systems
- Alternative way?
Theorem
Let \( V(x) = x^T P x, P = P^T \succ 0, P \) finite and define \( \chi_k(\tau) := x(t_k + \tau), \chi_k \in C([0, T], \mathbb{R}^n), k \in \mathbb{N}. \) Then the two following statements are equivalent:

(i) The LMI \( \Phi(T)^T P \Phi(T) - P \prec 0 \) holds for all \( T \in \mathcal{T}. \)

(ii) There exists a continuous functional \( V_1 : \mathbb{R} \times C([0, T], \mathbb{R}) \to \mathbb{R}, \) differentiable over \([t_k, t_{k+1})\) satisfying
\[
V_1(T_k, \chi_k) = V_1(0, \chi_k)
\]
for all \( k \in \mathbb{N} \) and such that the functional
\[
\mathcal{W}(\tau(t), \chi_k) := V(x(t)) + V_1(\tau(t), \chi_k(\tau(t)))
\]
satisfies
\[
\dot{\mathcal{W}}(\tau(t), \chi_k) = \frac{d}{dt} \mathcal{W}(\tau(t), \chi_k) < 0
\]
for all \( \tau \in [0, T_k], T_k \in \mathcal{T}, k \in \mathbb{N} - \{0\}. \)

Moreover, if one of these two statements is satisfied, the solutions of the sampled-data are asymptotically stable.
Illustration of the result
Connection with impulsive approach

- In the impulsive framework [Naghshtabrizi et al.], [Seuret], the functional may be considered

\[ V = x(t)^T P x(t) + (T_k - \tau)(x(t) - x(t_k))^T S(x(t) - x(t_k)) \]
\[ + (T_k - \tau) \int_{t_k}^{t} \dot{x}(s)^T R \dot{x}(s) ds, \quad t \in [t_k, t_{k+1}] \]

where \( \tau = t - t_k \) and \( P, S, R \) symmetric positive definite.

- When \( \tau = 0 \) and \( \tau = T_k \), the two last terms are 0

- Functional satisfies the boundary conditions \( \rightarrow S \) and \( R \) do not need to be positive definite.
Stability analysis
Proposed functional

- **Starting point [Seuret]**

\[
V = x(t)^T P x(t) + V_1(t)
\]
\[
V_1 = (T_k - \tau) \zeta(t)^T [S \zeta(t) + 2Q x(t_k)] + (T_k - \tau) \int_{t_k}^{t} \dot{x}(s)^T R \dot{x}(s) ds + (T_k - \tau) \tau x(t_k)^T U x(t_k)
\]
\[
\zeta(t) = x(t) - x(t_k)
\]

- **Fragmentation (Discretization) \(N\) pieces \((N + 1\) points)\)

\[
t^i_k(t) = t_k + \frac{N - i}{N} (t - t_k)
\]
\[
\int_{t_k}^{t} \dot{x}(s)^T R \dot{x}(s) ds \rightarrow \sum_{i=0}^{N-1} \int_{t_k}^{t^i_k(t)} \dot{x}(s)^T R \dot{x}(s) ds
\]
\[
\zeta(t) \rightarrow \zeta_k(t) = \text{col} \{ x(t^i_k(t)) - x(t^{i-1}_k(t)) \}_{i=0,\ldots,N-1}
\]
Stability condition

Theorem

The sampled-data system is asymptotically stable for any time-varying sampling period in \([T_{MIN},T_{MAX}]\) if there exist constant matrices \(P = P^T \succ 0\), \(R_i = R_i^T \succ 0\), \(i = 0, \ldots, N - 1\) and \(U = U^T \in \mathbb{R}^{n \times n}\), \(S = S^T \in \mathbb{R}^{nN \times nN}\), \(Q \in \mathbb{R}^{nN \times n}\) and \(Y \in \mathbb{R}^{n(N+1) \times nN}\) such that the LMIs

\[
\begin{bmatrix}
\Psi_1 + T_{MIN} (\Psi_2 + \Psi_3) & 0 \\
0 & \Psi_1 + T_{MAX} (\Psi_2 + \Psi_3)
\end{bmatrix} \prec 0,
\]

\[
\begin{bmatrix}
\Psi_1 - T_{MIN} \Psi_3 & T_{MIN} Y \\
* & -\alpha^- R
\end{bmatrix} \prec 0,
\]

\[
\begin{bmatrix}
\Psi_1 - T_{MAX} \Psi_3 & T_{MAX} Y \\
* & -\alpha^+ R
\end{bmatrix} \prec 0
\]

hold where \(\alpha^- = N T_{MIN}\), \(\alpha^+ = N T_{MAX}\).

- Affine in \(T \rightarrow\) semi-infiniteness easy to handle
- Affine in the system matrices \(\rightarrow\) easy to extend to uncertain system
- No state-transition matrix involved \(\rightarrow\) nonlinear systems
Example 1

Let us consider the system

\[ \dot{x}(t) = Ax(t) + BKx(t_k) \]

with

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} \]

Constant sampling-period \( T = [0, 1.7294] \).

<table>
<thead>
<tr>
<th>Theorems</th>
<th>Ex.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Fridman et al., 04]</td>
<td>[0, 0.869]</td>
</tr>
<tr>
<td>[Naghshtabrizi et al., 08]</td>
<td>[0, 1.113]</td>
</tr>
<tr>
<td>[Fridman et al., 10]</td>
<td>[0, 1.695]</td>
</tr>
<tr>
<td>[Liu et al., 09]</td>
<td>[0, 1.695]</td>
</tr>
<tr>
<td>Proposed result, ( N = 1 )</td>
<td>[0, 1.721]</td>
</tr>
<tr>
<td>Proposed result, ( N = 3 )</td>
<td>[0, 1.727]</td>
</tr>
<tr>
<td>Proposed result, ( N = 5 )</td>
<td>[0, 1.728]</td>
</tr>
</tbody>
</table>
Example 2

- Let us consider the system

\[ \dot{x}(t) = Ax(t) + BKx(t_k) \]

with

\[ A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \]

- Constant sampling-period \( T = [0, 3.2716] \)

<table>
<thead>
<tr>
<th>Theorems</th>
<th>Ex.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Fridman et al., 04]</td>
<td>[0, 0.99]</td>
</tr>
<tr>
<td>[Naghshtabrizi et al., 08]</td>
<td>[0, 1.99]</td>
</tr>
<tr>
<td>[Fridman et al., 10]</td>
<td>[0, 2.03]</td>
</tr>
<tr>
<td>[Liu et al., 09]</td>
<td>[0, 2.53]</td>
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<tr>
<td>Proposed result, ( N = 1 )</td>
<td>[0, 2.51]</td>
</tr>
<tr>
<td>Proposed result, ( N = 3 )</td>
<td>[0, 2.62]</td>
</tr>
<tr>
<td>Proposed result, ( N = 5 )</td>
<td>[0, 2.64]</td>
</tr>
</tbody>
</table>
Example 3

Let us consider the system

\[ \dot{x}(t) = Ax(t) + BKx(t_k) \]

with

\[ A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]

Constant sampling-period \( T = [0.2007, 2.0207] \)

<table>
<thead>
<tr>
<th>Theorems</th>
<th>Ex.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Fridman et al., 04]</td>
<td>-</td>
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<tr>
<td>[Naghshtabrizi et al., 08]</td>
<td>-</td>
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<tr>
<td>[Fridman et al., 10]</td>
<td>-</td>
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<tr>
<td>[Liu et al., 09]</td>
<td>-</td>
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<tr>
<td>Proposed result, ( N = 1 )</td>
<td>[0.40, 1.11]</td>
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<tr>
<td>Proposed result, ( N = 3 )</td>
<td>[0.40, 1.28]</td>
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<tr>
<td>Proposed result, ( N = 5 )</td>
<td>[0.40, 1.31]</td>
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</tbody>
</table>
Conclusion
Conclusion

- Functional-based approach suitable for stability analysis
- Fragmentation improves results
- Possible extension to uncertain and nonlinear systems
Thank you for your attention!