Memory Resilient Gain-Scheduled State-Feedback Control of LPV Time-Delay Systems with Time-Varying Delays

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Plan

- Introduction
- Stability of Time-Delay Systems
- Synthesis of Memory-Resilient Controllers
- Example

Introduction

- Class of systems
- Classes of Controllers
- Objectives

Class of systems

- LPV Time-Delay systems

$$
\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} &= \begin{bmatrix} A(\rho(t)) \\ C(\rho(t)) \end{bmatrix} x(t) + \begin{bmatrix} A_h(\rho(t)) \\ C_h(\rho(t)) \end{bmatrix} x(t-h(t)) + \begin{bmatrix} E(\rho(t)) \\ F(\rho(t)) \end{bmatrix} w(t) \\
x(\theta) &= \phi(\theta),\ \theta \in [-h_M, 0]
\end{aligned}
$$

- Parameters :

$$
\begin{aligned}
\rho(t) &\in U_\rho \\
\dot{\rho}(t) &\in co[U_\nu]
\end{aligned}
$$

- Delay :

$$
\begin{aligned}
h(t) &\in [0, h_M] \\
\dot{h}(t) &< \mu < 1
\end{aligned}
$$

- Stability ?
- Control ?

Classes of Controllers

- Memoryless Gain-Scheduled (GS) State-Feedback (SF) Controllers

$$u(t) = K_0(\rho)x(t)$$

- Exact-Memory GS SF Controllers

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

- Approximate-Memory GS SF Controllers

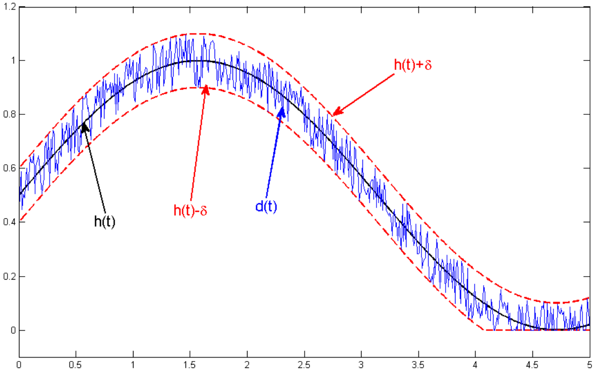
$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t))$$

where

$$|h(t) - d(t)| \leq \delta$$

Memoryless $\delta = h_M$ / Exact-memory $\delta = 0$

Delay Trajectories



Existing results

- Resilience w.r.t. gain uncertainties [Keel], [Peaucelle]
- Distributed delays [Mondie] [Gurdin and Mirkin]
- Resilience w.r.t. delay uncertainty (discrete delay) [Verriest], [Sename]
  - A posteriori
  - No guaranteed bounds
  - LTI case
  - Performance deterioration/instability

Objectives

- Wider class of systems
- Bounds guarantee/optimization
- Quantify performance degradation/Guarantee performance

Stability of Time-Delay Systems

- Lyapunov-Krasovskii Functional
- Stability Condition
- Associated Relaxation

Lyapunov-Krasovskii Functional

$$\begin{aligned} V(x_t, \dot{x}_t, \rho) &= V_1(x_t, \rho) + V_2(x_t) + V_3(\dot{x}_t) \\ V_1(x_t, \rho) &= x(t)^T P(\rho) x(t) \\ V_2(x_t) &= \int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta \\ V_3(\dot{x}_t) &= h_M \int_{-h_M}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta \end{aligned}$$

- Simple form
- Low computational complexity
- Possibility of extending it to a more general form

Theorem

The LPV time-delay system is asymptotically stable for all $h(t) \in [0, h_M]$ *such that* $\dot{h}(t) < \mu$ *if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^n_{++}$, *constant matrices* $Q, R \in \mathbb{S}^n_{++}$ *such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) & P(\rho)A_h(\rho) + R & h_M A(\rho)^T R \\ \star & -(1-\mu)Q - R & h_M A_h(\rho)^T R \\ \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where*

$\Psi(\rho,\nu) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \sum_i \nu_i \dfrac{\partial}{\partial \rho_i} P(\rho)$.

Bounded Real Lemma

Theorem

The LPV time-delay system is asymptotically stable for all $h(t) \in [0, h_M]$ such that $\dot{h}(t) < \mu$ if there exist a continuously differentiable matrix function $P : U_\rho \rightarrow \mathbb{S}^n_{++}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that the LMI

$$\begin{bmatrix} \Psi(\rho,\nu) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) & C(\rho)^T & h_M A(\rho)^T R \\ \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & h_M A_h(\rho)^T R \\ \star & \star & -\gamma I & F(\rho)^T & h_M E(\rho)^T R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ with

$\Psi(\rho,\nu) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \sum_i \nu_i \dfrac{\partial}{\partial \rho_i} P(\rho)$.

Moreover we have $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

- Difficult to apply for synthesis problems

Relaxed Bounded Real Lemma

Theorem

The LPV time-delay system is asymptotically stable for all $h(t) \in [0, h_M]$ *such that* $\dot{h}(t) < \mu$ *if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^n_{++}$*, a matrix function* $X : U_\rho \to \mathbb{R}^{n\times n}$*, constant matrices* $Q, R \in \mathbb{S}^n_{++}$ *and a scalar* $\gamma > 0$ *such that the LMI*

$$\begin{bmatrix} -(X+X^T) & P+X^TA & X^TA_h & X^TE & 0 & X^T & h_MR \\ \star & \Phi_1 & R & 0 & C^T & 0 & 0 \\ \star & \star & \Phi_2 & 0 & C_h^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P & -h_MR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ *with* $\Phi_1 = P + Q - R + \sum_i \nu_i \dfrac{\partial}{\partial \rho_i} P(\rho)$ *and* $\Phi_2 = -(1-\mu)Q - R$.
Moreover we have $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Controller Synthesis

- Closed-loop systems
- LMI Results

Closed-loop systems

- General closed-loop system

$$\dot{x}(t) = (A + BK_0)x(t) + (A_h + \alpha BK_h)x(t - h(t)) + \beta BK_h x(t - d(t))$$

- When
  - $\alpha = 0, \beta = 0 \Longrightarrow$ Memoryless control law
  - $\alpha = 1, \beta = 0 \Longrightarrow$ Exact-memory control law
  - $\alpha = 0, \beta = 1 \Longrightarrow$ Approximate-memory control law
- When $\alpha = 0, \beta = 1$ and
  - $\delta = 0$ (i.e. $d(t) = h(t)$) $\Longrightarrow$ Exact memory control law
  - $\delta = h_M$ (i.e. $d(t) \in [0, 2h_M]$) $\Longrightarrow$ Memoryless Control law
- Generalization of controller
- Single Framework

Memoryless and Exact Memory

Theorem

There exists an exact memory control law stabilizing LPV time-delay system for all $h(t) \in [0, h_M]$ *such that* $\dot{h}(t) < \mu$ *if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^n_{++}$*, matrix functions* $X : U_\rho \to \mathbb{R}^{n \times n}$*,* $Y_0, Y_h : U_\rho \to \mathbb{R}^{m \times n}$*, constant matrices* $Q, R \in \mathbb{S}^n_{++}$ *and a scalar* $\gamma > 0$ *such that the LMI*

$$\begin{bmatrix} -(X + X^T) & P + \mathcal{A} & \mathcal{A}_h & \mathcal{E} & 0 & X & h_M R \\ \star & \Phi_1 & R & 0 & \mathcal{C}^T & 0 & 0 \\ \star & \star & \Phi_2 & 0 & \mathcal{C}_h^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P & -h_M R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$*.*
Moreover the controller gains are given by $K_0 = Y_0 X^{-1}$*,* $K_h = Y_h X^{-1}$ *and we have* $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$*.*

Approximate Memory Control Law

- Closed-loop system

$$\dot{x}(t) = (A + BK_0)x(t) + A_h x(t - h(t)) + BK_h x(t - d(t)) + Ew(t)$$

- Systems with two interrelated delays

How to consider the relation $|d(t) - h(t)| \leq \delta$?

- Classical Lyapunov-Krasovskii for two delays will not apply
- Difficult to consider the trajectories of delays

Model Transformation

$$\nabla(z(t)) := \frac{1}{\delta}\int_{t-d(t)}^{t-h(t)} z(s)ds$$

- Properties
  - $||\nabla||_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$
  - $\nabla(\dot{x}(t)) = \frac{1}{\delta}(x(t-h(t)) - x(t-d(t)))$

$$x(t-h(t)) = x(t-d(t)) + \delta\nabla(\dot{x}(t))$$

- New closed-loop system

$$\begin{array}{rcl} \dot{x}(t) & = & (A+BK_0)x(t) + (A_h+BK_h)x(t-d(t)) + Ew(t) \\ & & +\delta A_h w_0(t) \\ z_0(t) & = & \dot{x}(t) \\ w_0(t) & = & \nabla(z_0(t)) \end{array}$$

- Uncertain system with a single delay $\Rightarrow$ Lyapunov-Krasovskii functional + scaled-small gain theorem

Lyapunov-Krasovskii functional

$$\begin{aligned}
V &= x(t)^T P(\rho)x(t) + \int_{t-h(t)}^{t} x(s)^T Q x(s) ds + h_M \int_{-h_M}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta \\
H &= V - \underbrace{\int_0^t \gamma w(s)^T w(s) - \gamma^{-1} z(s)^T z(s) ds}_{\text{performance}} - \underbrace{\int_0^t w_0(s)^T L w_0(s) - z_0(s)^T L z_0(s) ds}_{\text{robustness}} \\
L &= L^T \succ 0
\end{aligned}$$

$$x(t - h(t)) = x(t - d(t)) + \delta w_0(t)$$

Theorem

There exists an exact memory control law stabilizing LPV time-delay system for all $h(t) \in [0, h_M]$ such that $\dot{h}(t) < \mu$ if there exist a continuously differentiable matrix function $P : U_\rho \rightarrow \mathbb{S}^n_{++}$, $K_0, K_h : U_\rho \rightarrow \mathbb{R}^{m \times n}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that the BMI

$$\begin{bmatrix} \Psi & PA_{hcl} + R & \delta PA_h + \delta R & PE & C_{cl}^T & h_M A_{cl}^T R & A_{cl}^T L \\ \star & -Q_\mu - R & -\delta(Q_\mu + R) & 0 & C_{hcl}^T & h_M A_{hcl}^T R & A_{hcl}^T L \\ \star & \star & -\delta^2(Q_\mu + R) - L & 0 & \delta C_h^T & h_M \delta A_h^T R & \delta A_h^T L \\ \star & \star & \star & -\gamma I_p & F^T & h_M E^T R & E^T L \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -R & 0 \\ \star & \star & \star & \star & \star & \star & -L \end{bmatrix} \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ with $Q_\mu = (1 - \mu)Q$ and

$$\begin{aligned} \Psi &= A_{cl}^T P + PA_{cl} + \dot{P} + Q - R \\ A_{cl} &= A + BK_0 \qquad A_{hcl} = A_h + BK_h \\ C_{cl} &= C + DK_0 \qquad C_{hcl} = C_h + DK_h \end{aligned}$$

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Example (1)

- LPV time-delay system [Zhang et al., 2005]

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+0.1\rho(t) \\ -2 & -3+0.2\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho(t) \\ 0.1+0.1\rho(t) \end{bmatrix} u(t) \\
&\quad + \begin{bmatrix} 0.2\rho(t) & 0.1 \\ -0.2+0.1\rho(t) & -0.3 \end{bmatrix} x(t-h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\
z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \\
\rho(t) &= \sin(t)
\end{aligned}$$

Goal

- Find a controller such that such that the closed-loop system
  1. is asymptotically stable for all $h(t) \in [0, h_{max}]$ with $|\dot{h}(t)| \leq \mu < 1$ and
  2. satisfies

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

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Example (2)

Case 1 : $\dot{h}(t) \leq 0.5$, $h(t) \in [0, 0.5]$

- Design of a memoryless state-feedback control law

$$u(t) = K_0(\rho)x(t)$$

	minimal $\mathcal{L}_2$ gain
[Zhang et al. 2005]	$\gamma^* = 3.09$
[Briat et al. IFAC WC 2008]	$\gamma^* = 2.27$
This result	$\gamma^* = 1.90$

$$K_0(\rho) = \begin{bmatrix} -1.0535 - 2.9459\rho + 1.9889\rho^2 \\ -1.1378 - 2.6403\rho + 1.9260\rho^2 \end{bmatrix}^T$$

- Better performances
- Lower controller gains
- Lower numerical complexity

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Example (3)

Case 2 : $\dot{h}(t) \leq 0.9$, $h(t) \in [0, 10]$

- Synthesis of both memoryless and exact memory controllers

$$u(t) = K_0(\rho)x(t) \qquad u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

	minimal $\mathcal{L}_2$ gain
Memoryless Controller	$\gamma^* = 12.8799$
Exact Memory Controller	$\gamma^* = 4.1641$

+ Delayed term important
− Needs the exact value of the delay at any time
− Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay

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Example (4)

- Previous results : $\gamma = 12.8799$ (Memoryless), $\gamma = 4.1641$ (Exact Memory)

Memory Resilient synthesis

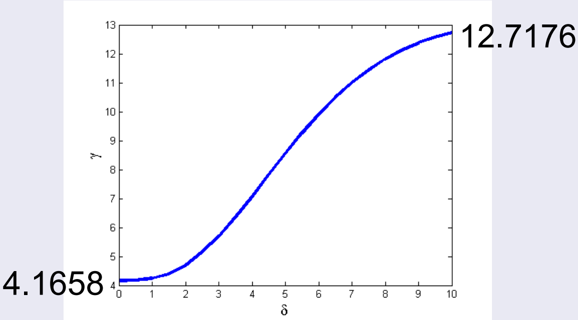

FIGURE: Best $\mathcal{L}_2$ performance γ vs. maximal error uncertainty δ

- Characterization of intermediate performances
- Direct generalization of the previous approach

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Memory Resilient synthesis

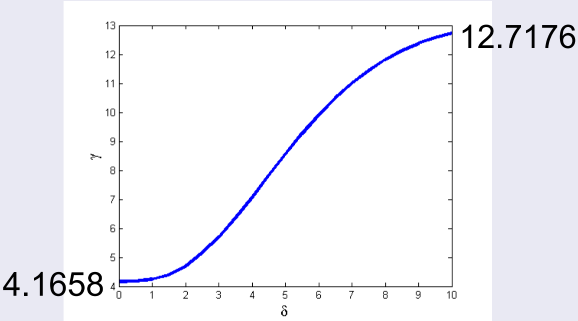

FIGURE: Best $\mathcal{L}_2$ performance γ vs. maximal error uncertainty δ

- Characterization of intermediate performances
- Direct generalization of the previous approach

Conclusion

- Resilience w.r.t. delay uncertainty
- Time-varying systems and delays
- Optimization : performance, maximal delay uncertainty
- Constructive

Future Work

- More general functionals
- Robustness
- Relaxation : tighter, gap

Thank you for your attention

תודה על תשומת לב