Stability analysis of asynchronous sampled-data systems with discrete-time constant input delay

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Alexandre Seuret[1], Corentin Briat[2] and Frédéric Gouaisbaut[1]

[1] CNRS, LAAS-UPS, Univ Toulouse, France
[2] D-BSSE, Swiss Federal Institute of Technology Zurich, Switzerland

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Outline

Preliminaries

Consider linear continuous-time systems of the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \; t \geq 0, \\ x(0) &= x_0 \end{cases} \tag{1}$$

The control input u obeys the following equation :

$$u(t) = Kx(t_{k-h}), \; t \in [t_k, t_{k+1}), \; k \in \mathbb{N}, \tag{2}$$

⋆ $K \in \mathbb{R}^{m \times n}$ is the controller gain.
⋆ $h \in \mathbb{N}$ is constant value modeling a discrete delay.
⋆ The sequence $\{t_k\}_{k \in \mathbb{N}}$, $t_0 = 0$, is the sequence of sampling instants satisfying:

$$T_k := t_{k+1} - t_k \in [T_{min}, T_{max}], \forall k \in \mathbb{N};$$

$$0 < T_{min} \leq T_{max} < \infty$$

closed loop systems

The closed-loop system obtained from the interconnection of (1) and (2) is given by

$$\begin{cases} \dot{x}(t) = Ax(t) + BKx(t_{k-h}), \; t \in [t_k, t_{k+1}), \; k \in \mathbb{N}, \\ x(t_{-\theta}) = \psi(\theta), \quad \theta = 0, \dots, h \end{cases} \tag{3}$$

⋆ initial conditions $\psi(\theta) \in \mathbb{R}^n$ with $\psi(0) = x_0$

⋆ $\{t_{-h}, \dots, t_{-1}\}$ is a sequence of negative real numbers.

objectives: Stability analysis of (uncertain) aperiodic sampled-data systems with discrete-time input delay using a looped-functional based approach.

Stability analysis

Several approaches have been proposed

- Discrete time approaches [Oishi09, Hetel06, Cloosterman09].

$$x(t_{k+1}) = e^{AT_k}x(t_k) + \int_0^{T_k} e^{As}BK\mathrm{d}s x(t_{k-h}), \; k \in \mathbb{N}, \tag{4}$$

- Input delay approach [Fridman04,10; Seuret09] $\rightarrow$ modeling the original system into an input time delay and use of adequate LKF.
- Robust analysis techniques based on IQCs, Small gain [Mirkin07, Fujioka09, Kao13, Briat11] $\rightarrow$ merge the closed loop system into a nominal system connected with some elements depending on the sampling process.
- Impulsive systems $\rightarrow$ functionals [Naghshtabrizi11], clock dependent Lyapunov functions [Briat13].

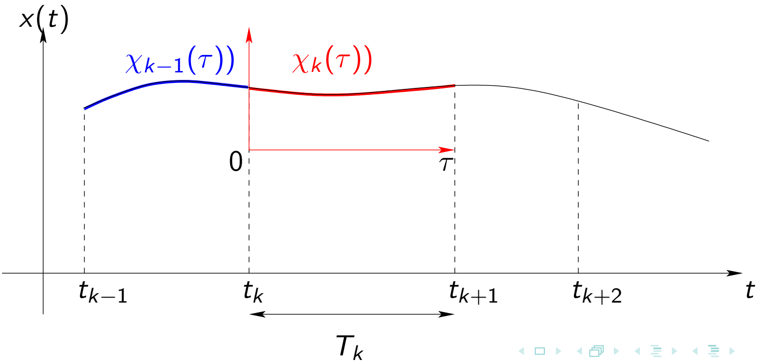
In this present paper, we address the problem by using the looped functional approach.

The looped-functional-based approach

⋆ As depicted, characterization of the trajectories of (3) in a lifted domain.

⋆ The entire state-trajectory viewed as a sequence of functions $\{x(t_k+\tau),\ \tau \in (0, T_k]\}_{k\in\mathbb{N}}$, ie

$$\chi_k(\tau) := x(t_k+\tau) \quad \text{with} \quad \chi_k(0) = \lim_{s\downarrow t_k} x(s).$$



Modeling the closed loop in the lifted domain

⋆ The system (3) rewrites then as

$$\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_{k-h}(0), \tau \in [0, T_k), k \in \mathbb{N}. \qquad (5)$$

Introducing the notation for the delayed state :

$$\chi_k^h(\theta) = \chi_{k+\theta}(0) = x(t_{k+\theta}), \theta \in \{-h, -h+1, \dots, 0\}$$

equivalently, we obtain also:

$$\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_k^h(-h), \tau \in [0, T_k), k \in \mathbb{N}. \qquad (6)$$

We also denote by χ_k^h the vector $\mathrm{col}(\chi_k^h(0), \chi_k^h(-1), \dots, \chi_k^h(-h))$.

Let us first define looped-functionals

Looped functionals

Let $0 < T_{min} \leq T_{max} < \infty$. A functional

$$f : [0, T_{max}] \times \mathbb{K} \times [T_{min}, T_{max}] \to \mathbb{R},$$

where $\mathbb{K} := \bigcup_{T \in [T_{min},\ T_{max}]} \mathcal{C}([0,\ T] \to \mathbb{R}^n$, is said to be a looped-functional if

1. the equality
$$f(0, z, T) = f(T, z, T) \tag{7}$$
holds for all functions $z \in \mathbb{K}$ and all $T \in [T_{min},\ T_{max}]$ and
2. it is differentiable with respect to the first variable with the standard definition of the derivative.

Theorem

Let $V : \mathbb{R}^n \times \mathbb{R}^{n(h+1)} \to \mathbb{R}_+$ be a quadratic form verifying $\mu_1|\varphi_2|^2 \leq V(\varphi_1, \varphi_2) \leq \mu_2|\varphi_2|^2$ for $0 < \mu_1 \leq \mu_2$. Assume that one of the following equivalent statements hold:

1. *The sequence $\{V(\chi_k(0), \chi_k^h)\}_{k\in\mathbb{N}}$ is decreasing.*
2. *There exists a looped-functional $\mathcal{V}$ such that*

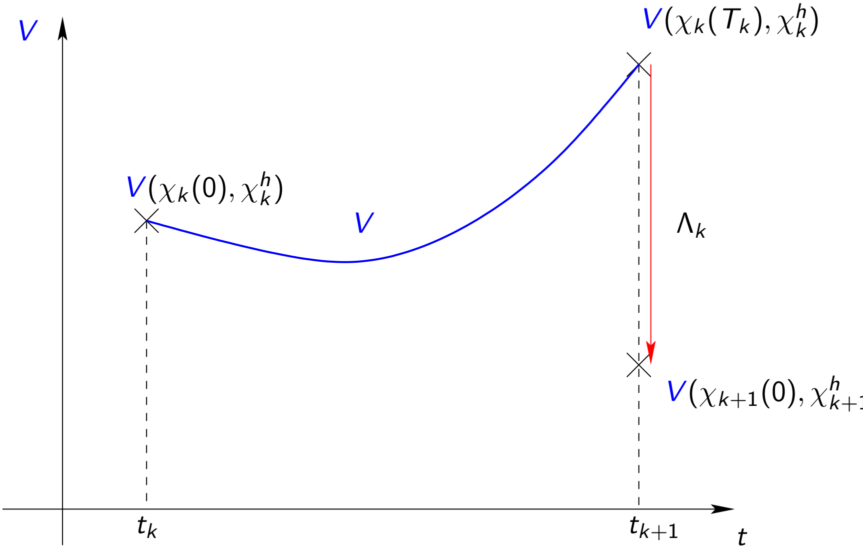
$$\mathcal{W}_k(\tau, \chi_k, \chi_k^h) := \frac{\tau}{T_k}\Lambda_k + V(\chi_k(\tau), \chi_k^h) + \mathcal{V}(\tau, \chi_k, T_k) \tag{8}$$

with $\Lambda_k = V(\chi_{k+1}(0), \chi_{k+1}^h) - V(\chi_k(T_k), \chi_k^h)$ is decreasing along the trajectories of the system (5), $\forall \tau \in [0, T_k]$, $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

Then, the system (3) with delay h is asymptotically stable for any sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfying $t_{k+1} - t_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

$\star$ The proof is similar to the one derived in [Seuret10,Briat11].

How does it work?



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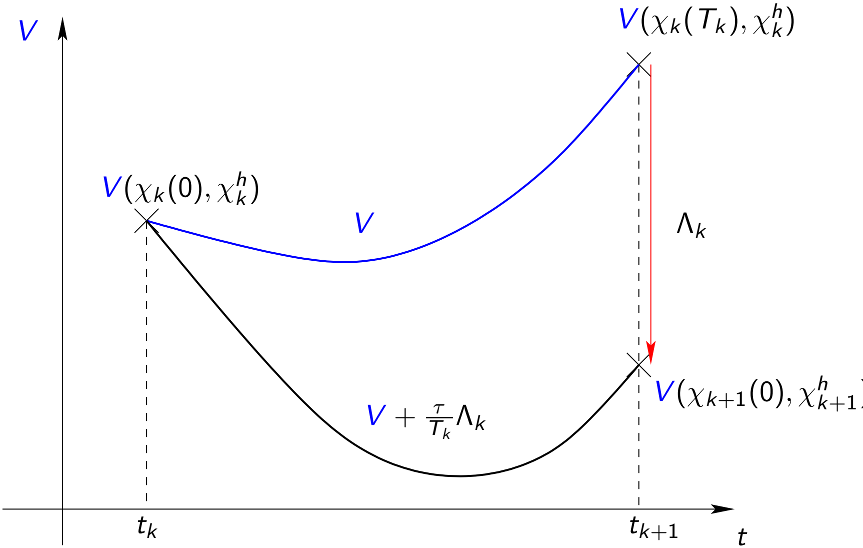

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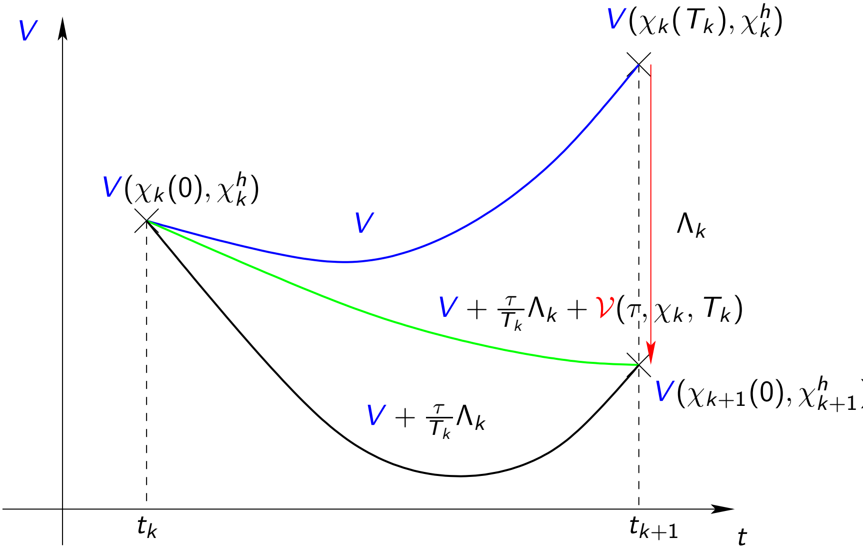

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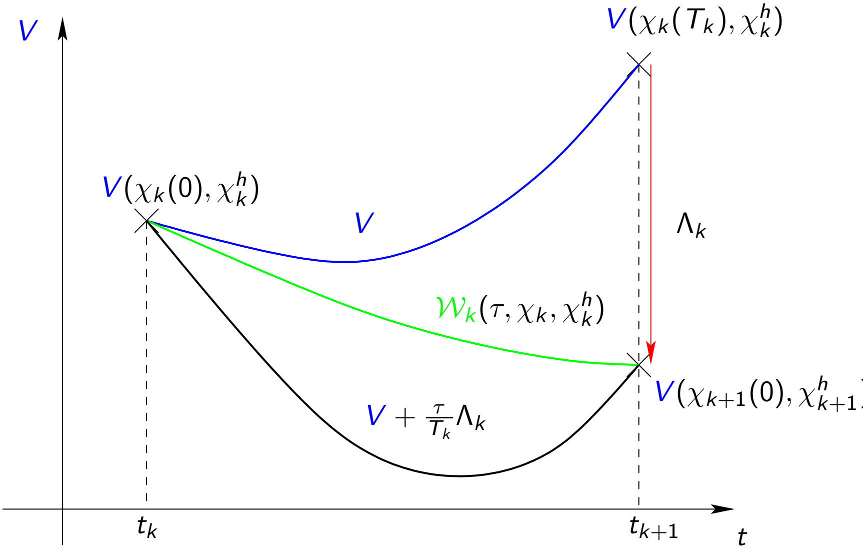

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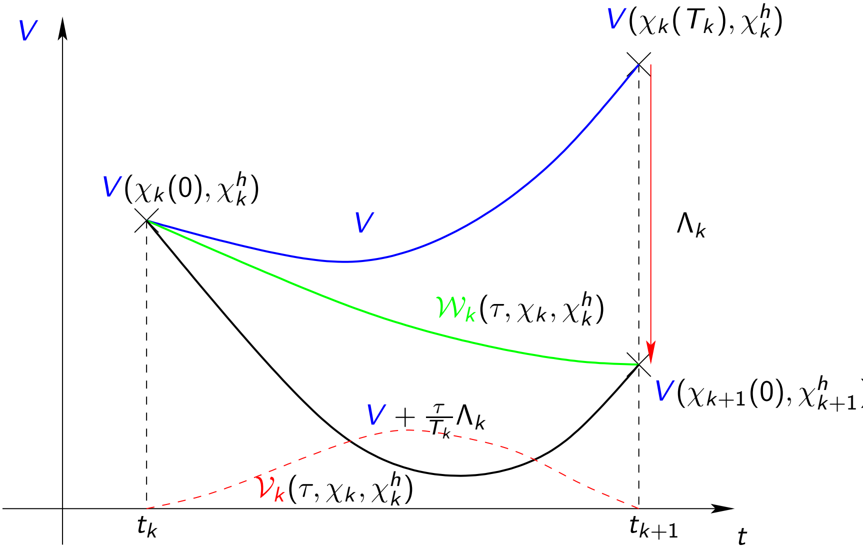

Remarks

$\star$ The function $V(\chi_k(0), \chi_k^h)$ constitutes a Lyapunov function for the discrete time system.

$$x(t_{k+1}) \quad = \quad e^{AT_k}x(t_k) + \int_0^{T_k} e^{As}BK\mathrm{d}s x(t_{k-h}), \ k \in \mathbb{N},$$

$\star$ The looped functional does not need to positive definite but must satisfy the boundary conditions.

Methodology

- Select a classical function V,
- Select a looped functional $\mathcal{V}$ with the boundary conditions satisfied by construction,
- Construct $\mathcal{W}_k$ and prove that the derivative along the system (5) is negative definite,
- Find a way to get some LMI conditions.

Theorem

The sampled-data system (5) with the delay h is asymptotically stable for all $T_k := t_{k+1} - t_k \in [T_{min},\ T_{max}]$, $k \in \mathbb{N}$, *if there exist matrices* $R, Q, Z \in \mathbb{S}^n_+$, $P, X\ in \mathbb{S}^{2n}$, $S_1, S_2 \in \mathbb{S}^n$, $U_1, U_2 \in \mathbb{R}^{n\times n}$ *such that a given set of LMIs*$(R, Q, Z, P, X, S_1, S_2, U_1, U_2, T_k)$ *are satisfied*

⋆ The result is expressed in terms of LMIs (described in the paper).

⋆ The LMis condition is linear with respect to T_k allowing to test the condition of the vertices of the interval T_{min} and T_{max}.

Proof(1)

1 Selection of the Lyapunov function for the discrete-time system:

$$V(\chi_k(\tau),\chi_k^h) = \begin{bmatrix}\chi_k(\tau)\\ \chi_k(0)\end{bmatrix}^{\mathsf{T}} P \begin{bmatrix}\chi_k(\tau)\\ \chi_k(0)\end{bmatrix} + \sum_{i=k-h}^{k-1} \chi_i^{\mathsf{T}}(0) Q \chi_i(0) + h \sum_{i=-h}^{-1} \sum_{j=k+i}^{k-1} \delta_i^{\mathsf{T}}(0) R \delta_i(0)$$

where $\delta_i(0) = \chi_{i+1}(0) - \chi_i(0)$.

- Extension of the classical Lyapunov functional for discrete time delay system (for $\tau = 0$, we obtain a LKF for the discrete time delay system).
- $V(\chi_k(0),\chi_k^h) > 0$ if $[I,\ I]P[I,I]^T$, Q and R are also positive definite.
- This functional, defined on the interval $[t_k, t_{k+1})$, is discontinuous from an interval to another due to the terms $\chi_i(0), i \in \{k-h, k\}$.

2 Selection of the Looped functional:

$$
\begin{aligned}
T_k \mathcal{V}(\tau, \chi_k, T_k) \quad = \quad & \begin{bmatrix} \chi_k(\tau_k) \\ \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^T M(\tau, Z, X, S_i, U_i, T_k) \begin{bmatrix} \chi_k(\tau_k) \\ \chi_k(\tau) \\ \chi_k(0) \end{bmatrix} \\
& + T_k^2 \int_0^{\tau} \dot{\chi}_k(s)^{\intercal} Z \dot{\chi}_k(s) ds \\
& - T_k \tau \int_0^{T_k} \dot{\chi}_k(s)^{\intercal} Z \dot{\chi}_k(s) ds
\end{aligned} \tag{9}
$$

where $M(.)$ is an adequate matrix depending on τ and some decisions variables Z, X, S_i, U_i, T_k.

- This matrix is chosen such that

$$\mathcal{V}(0, \chi_k, T_k) = \mathcal{V}(T_k, \chi_k, T_k) = 0$$

for all $T_k \in [T_{min}, \ T_{max}]$.

Proof (3)

Following Theorem 1, we consider

$$\dot{\mathcal{W}}_k(\tau,\chi_k,\chi_k^h) \;=\; \frac{1}{T_k}\Big(\Lambda_k + T_k\dot{V}(\chi_k(\tau),\chi_k^h) + T_k\dot{\mathcal{V}}(\tau,\chi_k,T_k)\Big)$$

where Λ_k is defined in Theorem 1.

⋆ We calculate $\dot{\mathcal{W}}_k(\tau,\chi_k,\chi_k^h)$ along the trajectories of (5).

⋆ The resulting condition has to be modified to get a numerical tractable conditions:

- The Jensen inequality → bounds on the integrals.
- Park 's lemma → gets a convex condition with respect to $T_k \in [T_m, T_M]$ and decision variables.

An example with a periodic sampling

Consider the sampled-data system (3) with matrices:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 3.75, 11.5 \end{bmatrix}. \qquad (10)$$

Table : Maximal stability-preserving periodic sampling period $\bar{T} = T_{max} = T_{min}$ for the sampled-data system (3)-(11) for different values of the delay h.

h	0	1	2	5	10
Theoretical bounds	1.729	0.763	0.463	0.216	0.112
[Naghshtabrizi10] ($\tau = hT$)	1.278	0.499	0.333	0.166	0.090
[LiuFridman12] ($\tau = hT$)	1.638	0.573	0.371	0.179	0.096
[Seuret2011] ($\tau = hT$)	1.721	0.701	0.431	0.197	0.103
Theorem 2	1.728	0.761	0.448	0.199	0.103

The same example with an aperiodic sampling

Consider the sampled-data system (3) with matrices:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 3.75, 11.5 \end{bmatrix}. \qquad (11)$$

Table : Maximal stability-preserving T_{max} (with $T_{min} = 10^{-2}$) for the aperiodic sampled-data system (3)-(11) for different values of the delay h.

h	0	1	2	5	10
Theorem 2	1.708	0.618	0.377	0.176	0.094

What we presented

- A novel way for analyzing stability of periodic and aperiodic uncertain sampled-data systems with discrete-time delays,
- Approach based on a looped functional and Lyapunov-Krasovskii functionals.

What we would like to do

- Extension to the case of time varying delay,
- Use of tighter inequalities in order to reduce the overall conservatism of the approach.