Robust stability analysis in the $*$-norm and Lyapunov-Razumikhin functions

Corentin Briat

KTH, Stockholm, Sweden

December 14^{th} 2011
CDC 2011, Orlando, USA

Outline

- Introduction
- I/O approaches and robust stability analysis
- Application to time-delay systems
- Conclusion and Ideas

Introduction

Problem statement

Time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bx(t - h(t)) \\ x(s) & = & \phi(s),\ s \in \mathcal{S} \subset \mathbb{R}_- \end{array}$$

- Delay may be bounded/unbounded
- 'No constraint' on the derivative
- Does not mean $\dot{h}$ can be 'infinite' !

Delay derivative problem

- Razumikhin condition for DIS can conclude on asymptotic stability even though the system is only stable!
- Example for $h(t) = t + 1$:

$$\dot{x}(t) = Ax(t) + Bx(-1)$$

Stability analysis tools

Lyapunov-based techniques

- Lyapunov-Krasovskii
- Lyapunov-Razumikhin

Input/Output Approaches

- Small-gain results and generalizations
- IQC-based techniques
- Well-posedness approach

I/O approaches and robust stability analysis

Linear Fractional Transformation

Uncertain system

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}(\nabla)x(t) \\ \nabla & \in & \boldsymbol{\nabla} \end{array}$$

Linear Fractional Representation

$$H : \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ew(t) \\ z(t) & = & Cx(t) + Fw(t) \end{array} \right.$$

$$w(t) = \Delta(\nabla)z(t)$$

- Input/output approaches for stability analysis
- What norm ?

Stability analysis

Small-gain

- Assume $||\Delta|| \leq 1$ then the interconnection is stable if

$$\left|\left|\begin{bmatrix} A & E \\ C & F \end{bmatrix}\right|\right| < 1.$$

Scaled-Small Gain

- Assume $||\Delta|| \leq 1$ then the interconnection is stable if there exists a matrix S such that $\Delta S = S\Delta$ and

$$\left|\left|\begin{bmatrix} A & ES \\ S^{-1}C & S^{-1}FS \end{bmatrix}\right|\right| < 1.$$

Scaled Small-gain, L_2-norm and LKF

$\Delta = \exp^{-sh}$

- $||S(sI - A)^{-1}BS^{-1}||_{H_\infty} < 1$ (convex problem)
- Equivalent to the LKF

$$V = x(t)^T Px(t) + \int_{t-h}^{t} x(s)^T S^T S x(s) ds$$

$\Delta_1 = e^{-sh}$ **and** $\Delta_2 = (1 - e^{-sh})/sh$

- Leads to the same conditions than the LKF

$$V = x(t)^T Px(t) + \int_{t-h}^{t} x(s)^T S_1^T S_1 x(s) ds + \int_{-h}^{0} \int_{t+s}^{t} \dot{x}(\theta)^T S_2^T S_2 \dot{x}(\theta) d\theta ds$$

- Also true for time-varying delays
- Other examples can be found
- What about other norms?

L_∞ and QL_∞-norms

Norms definition

- L_∞: $||w||_{L_\infty} := \text{ess}\sup\limits_t ||w(t)||_\infty$
- QL_∞: $||w||_{QL_\infty} := \sup\limits_t \sqrt{w(t)^T w(t)}$

Properties

- L_∞-norm does not always define the delay operator as a bounded operator
- $L\infty \subset QL_\infty$
- Upper-bound on the QL_∞-norm tractable
- $*$-norm: QL_∞-induced norm

$$||H||_* := \sup_{w \in QL_\infty,\ w \neq 0} \left\{ \frac{||Hw||_{QL_\infty}}{||w||_{QL_\infty}} \right\}$$

Example

$$
\begin{aligned}
w(t) &= \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{otherwise} \end{cases} \\
h(t) &= \begin{cases} 0 & \text{if } t \in [0, t_0] \\ t - t_0 & \text{otherwise} \end{cases} \\
w(t - h(t)) &= \begin{cases} 1 & \text{if } t \geq t_0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

norm	$\|\|w\|\|$	$\|\|\mathscr{D}_h(w)\|\|$	$\|\|\mathscr{D}_h\|\|$
ess sup $\|\| \cdot \|\|_\infty$	0	1	$+\infty$
QL_∞	1	1	1

Scaled $*$-Small-Gain

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ew(t) \\ z(t) & = & Cx(t) + Fw(t) \\ \hline w(t) & = & \Delta z(t) \end{array}$$

Theorem (Scaled Small $*$-Gain Theorem)

The uncertain system with $||\Delta||_* \leq \eta^{-1}$ is asymptotically stable if there exist symmetric matrices $P \succ 0$, $S \succ 0$ (with $\Delta S = S\Delta$) and scalars $\varepsilon, \xi > 0$ such that the matrix inequality

$$\begin{bmatrix} A^T P + PA + \xi P + \varepsilon I & PE & 0 & 0 \\ \star & -\eta S & 0 & F^T S \\ \star & \star & -\xi P & C^T S \\ \star & \star & \star & -\eta S \end{bmatrix} \preceq 0 \qquad (1)$$

holds.

Application to time-delay systems

Operator norms

Operator $\mathscr{D}_h$

- Delay operator: $\mathscr{D}_h(w)(t) = w(t - h(t))$
- $*$-norm given by:

$$||\mathscr{D}_h||_* = 1$$

- Note that $||\mathscr{D}_h||_{L_2 - L_2} = (1-\mu)^{-1/2}$ where $\dot{h} \leq \mu$
- $*$-norm does not depend on the delay derivative

Operator $\mathscr{S}_h$

- Integral-delay operator: $\mathscr{D}_h(w)(t) = \int_{t-h(t)}^{t} w(s)ds$
- $*$-norm given by:

$$||\mathscr{S}_h||_* = \bar{h}$$

- Note also that $||\mathscr{S}_h||_{L_2 - L_2} = \bar{h}$
- Both norms depend on the maximal delay value

DIS Stability - $\mathscr{D}_h$-operator

Linear Fractional Representation

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{D}_h(z)(t) \\ & = & x(t-h(t)) \end{array}$$

Stability condition

- The TDS is asymptotically stable independently of the delay if there exist a matrix $P = P^T \succ 0$ and a scalar $\xi > 0$ such that the matrix inequality

$$\begin{bmatrix} A^T P + PA + \xi P & PB \\ \star & -\xi P \end{bmatrix} \prec 0$$

  holds.
- Exactly the DIS Razumikhin condition!
- The L_2-norm would lead to the DIS Krasovskii condition

DDS Stability - $\mathscr{D}_h$- and $\mathscr{S}_h$-operators

Linear Fractional Representation

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \hline z(t) \end{bmatrix} = \left[\begin{array}{c|ccc} A+B & 0 & -\bar{h}BA & -\bar{h}B^2 \\ \hline I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \begin{bmatrix} \tilde{x}(t) \\ \hline w(t) \end{bmatrix}$$
$$w = \mathrm{diag}(\mathscr{D}_h, \mathscr{S}_h, \mathscr{S}_h)(z).$$

Stability condition

- The TDS is asymptotically stable for all $h(t) \in [0, \bar{h}]$, $\bar{h} > 0$ if there exist symmetric matrices $P, S_1 \succ 0$ and a scalar $\xi > 0$ such that the matrix inequality

$$\begin{bmatrix} (A+B)^T P + P(A+B) + \xi P & -\bar{h}PBA & -\bar{h}PB^2 \\ \star & -\xi P + S_1 & 0 \\ \star & \star & -S_1 \end{bmatrix} \prec 0$$

  holds.
- Simple DDS stability by Razumikhin is a corollary

DDS Stability - $\mathscr{D}_h$- and $\mathscr{S}_h$-operators

Linear Fractional Representation

$$\begin{aligned}\begin{bmatrix}\dot{\tilde{x}}(t)\\ \hline z(t)\end{bmatrix} &= \left[\begin{array}{c|cc} A+B & 0 & -\bar{h}B \\ \hline I & 0 & 0 \\ A & B & 0\end{array}\right]\begin{bmatrix}\tilde{x}(t)\\ \hline w(t)\end{bmatrix}\\ w &= \text{diag}(\mathscr{D}_h,\mathscr{S}_h)(z).\end{aligned}$$

Stability condition

- The TDS is asymptotically stable for all $h(t)\in[0,\bar{h}]$, $\bar{h}>0$ if there exist symmetric matrices $P, S_1, S_2 \succ 0$ and a scalar $\xi > 0$ such that the matrix inequalities

$$\begin{bmatrix} (A+B)^TP + P(A+B) + \xi P & -\bar{h}PB \\ \star & -S_2 \end{bmatrix} \prec 0$$

and

$$\begin{bmatrix} -S_1 + B^TS_2B & B^TS_2A \\ \star & -\xi P + S_1 + A^TS_2A \end{bmatrix} \preceq 0$$

hold.

Conclusion

Conclusions

Summary

- Razumikhin approach interpreted in the input/output framework
- QL_∞-norm is 'Razumikhin's norm'
- L_2 is 'Krasovskii's norm'
- New Razumikhin can be obtained
- Some of these are difficult (not possible) to obtain using the usual

Future works

- Improve Razumikhin ?
- Other operators (fragmented operators $\mathscr{D}_{ih/N}$, $i = 1, \ldots, N$.

Thank you for your attention