Robust stability analysis of uncertain Linear Positive Systems via Integral Linear Constraints: L_1 and L_∞-gain characterizations

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Outline

- Introduction
- Stability analysis and norm computation
- Robust stability analysis
- Conclusion and Future Works

Introduction

Linear positive systems

Internally positive systems

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ x(0) & = & x_0 \end{array} \quad (1)$$

- (P_1) Positive orthant $\mathbb{R}^n_+$ invariant: $x_0 \in \mathbb{R}^n_+ \Rightarrow x(t) \in \mathbb{R}^n_+$, for all $t \geq 0$
- NSC: A is a Metzler matrix (nonnegative off-diagonal elements)

Input/Output Positive systems

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ew(t) \\ z(t) & = & Cx(t) + Fw(t) \\ x(0) & = & x_0 \end{array} \quad (2)$$

- (P_1) holds
- (P_2) For all $w(t) \geq 0$, we have $z(t) \geq 0$
- NSC: A is Metzler and E, C, F are nonnegative

Stability analysis

Quadratic Lyapunov Functions

- $V(x) = x^T P x$, $P = P^T \succ 0$
- Enough to pick a diagonal P

$$A^T P + PA \prec 0$$

- Semidefinite programming, LMIs
- Suitable for L_2-gain analysis (H_∞-norm)

Copositive linear Lyapunov Functions

- $V(x) = \lambda^T x$, $\lambda > 0$

$$\lambda^T A < 0$$

- Linear programming
- Suitable for L_1- and L_∞-gain analysis

Induced norms for positive systems

- $h(t) \in \mathbb{R}_+^{q\times p}$: impulse response of the positive system Σ
- Output $z = h * w$ nonnegative when w nonnegative

L_1-norm and L_1-gain

$$||w||_{L_1} := \int_0^\infty \mathbb{1}^T w(s)ds \qquad ||\Sigma||_{L_1-L_1} := \max_j \left\{ \sum_i \int_0^\infty h_{ij}(s)ds \right\}$$

L_∞-norm and L_∞-gain

$$||w||_{L_\infty} := \operatorname{ess\,sup}_t ||w(t)||_\infty, \qquad \begin{aligned} ||\Sigma||_{L_\infty-L_\infty} &:= \max_i \left\{ \sum_j \int_0^\infty h_{ij}(s)ds \right\} \\ &= ||\Sigma^*||_{L_1-L_1} \end{aligned}$$

where

$$\Sigma^* = \left[\begin{array}{c|c} A^T & C^T \\ \hline E^T & F^T \end{array}\right].$$

Stability analysis and norm computation

L_1-gain analysis

Theorem

Let (A, E, C, F) be an input-output positive system. The following statements are equivalent:

1. *The system is asymptotically stable and the L_1-gain smaller than $\gamma > 0$*
2. *$G(s)$ is asymptotically stable and $\mathbb{1}_q^T G(0) < \gamma \mathbb{1}_q^T$*
3. *$G(s)$ is asymptotically stable and $\mathbb{1}_q^T (F - CA^{-1}E) < \gamma \mathbb{1}_q^T$*
4. *There exists a vector $\lambda > 0$ such that the inequalities*
   a. $\lambda^T A + \mathbb{1}_q^T C < 0$
   b. $\lambda^T E - \gamma \mathbb{1}_p^T + \mathbb{1}_q^T F < 0$

   hold.

Remarks

- Linear programming problem
- Actual L_1-gain retrieved by minimizing $\gamma > 0$

L_∞-gain analysis

Theorem

Let (A, E, C, F) be an input-output positive system. The following statements are equivalent:

1. *The system is asymptotically stable and the L_∞-gain smaller than $\gamma > 0$*
2. *$G(s)$ is asymptotically stable and $G(0)\mathbb{1}_p < \gamma\mathbb{1}_p$*
3. *$G(s)$ is asymptotically stable and $(F - CA^{-1}E)\mathbb{1}_p < \gamma\mathbb{1}_p$*
4. *There exists a vector $\lambda > 0$ such that the inequalities*
   - a. $A\lambda + E\mathbb{1}_p < 0$
   - b. $C\lambda - \gamma\mathbb{1}_q + F\mathbb{1}_p < 0$

   hold.

Remarks

- Linear program
- Convenient for control

Robust stability analysis

Uncertain systems and LFT

Uncertain system

$$\begin{array}{rcl} \dot{x}(t) & = & A_u(\delta)x(t) + E_u(\delta)w_1(t) \\ z_1(t) & = & C_u(\delta)x(t) + F_u(\delta)w_1(t) \\ \delta & \in & \boldsymbol{\delta} := [0,1]^N \end{array} \quad (3)$$

- $A_u(\delta)$ Metzler for all $\delta \in \boldsymbol{\delta}$
- $E_u(\delta), C_u(\delta)$ and $F_u(\delta)$ nonnegative for all $\delta \in \boldsymbol{\delta}$

Linear Fractional Representation

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + E_0w_0(t) + E_1w_1(t) \\ z_0(t) & = & C_0x(t) + F_{00}w_0(t) + F_{01}w_1(t) \\ z_1(t) & = & C_1x(t) + F_{10}w_0(t) + F_{11}w_1(t) \\ w_0(t) & = & \Delta(\delta)z_0(t) \end{array} \quad (4)$$

- A Metzler
- C_0, C_1, E_1, F_{01} and F_{11} nonnegative

Integral linear constraints

- $\Sigma \in \boldsymbol{\Sigma}$ where $\boldsymbol{\Sigma}$ is a family of positive operators
- $z = \Sigma w$ for some nonnegative input signal w
- Family can be characterized in terms of an Integral Linear Constraint (ILC)

$$\int_0^{\infty} \varphi_1^T z(s) + \varphi_2^T w(s) ds \geq 0 \qquad (5)$$

  for all $z = \Sigma w$, $\Sigma \in \boldsymbol{\Sigma}$.
- Scaling factors φ_1 and φ_2 chosen accordingly
- Frequency domain interpretation

$$\begin{array}{c} \varphi_1^T \widehat{z}(0) + \varphi_2^T \widehat{w}(0) \geq 0 \\ \Updownarrow \\ \left[\varphi_1^T \widehat{\Sigma}(0) + \varphi_2^T\right] \widehat{w}(0) \geq 0 \\ \Updownarrow \\ \varphi_1^T \widehat{\Sigma}(0) + \varphi_2^T \geq 0 \end{array} \qquad (6)$$

- Only $\omega = 0$ is important
- Last inequality contains parametric uncertainties only

Examples

Constant/Time-varying parameter uncertainty

- Parametric uncertainty $\delta(t) \in [0,1],\ t \geq 0$

$$\int_0^\infty \varphi^T(1-\delta(\theta))w(\theta)d\theta \geq 0 \Longleftrightarrow \varphi \geq 0$$

Constant delay operator

- $z(t) = w(t-h),\, \widehat{z}(s) = e^{-sh}\widehat{w}(s)$

$$\varphi_1^T\widehat{\Sigma}(0) + \varphi_2^T \geq 0 \Longleftrightarrow \varphi_1^T + \varphi_2^T \geq 0$$

Uncertain positive LTI system

- Uncertain asymptotically stable positive transfer function $H \in \mathscr{H}$
- Static gain $H(0) \in \mathscr{H}_0$

$$\varphi_1^T Z + \varphi_2^T \geq 0,\ Z \in \mathscr{H}_0 \tag{7}$$

Robust stability conditions

Theorem

The uncertain linear positive system is asymptotically stable if there exist $\lambda \in \mathbb{R}^n_{++}$, $\varphi_1(\delta), \varphi_2(\delta) \in \mathbb{R}^{n_0}$ *and* $\gamma > 0$ *such that the robust linear program*

$$\begin{array}{rcl} \lambda^T A + \varphi_1(\delta)^T C_0 + \mathbb{1}_q^T C_1 & < & 0 \\ \lambda^T E_0 + \varphi_2(\delta)^T + \varphi_1(\delta)^T F_{00} + \mathbb{1}_q^T F_{10} & < & 0 \\ \lambda^T E_1 - \gamma \mathbb{1}_p^T + \varphi_1(\delta)^T F_{01} + \mathbb{1}_q^T F_{11} & < & 0 \end{array} \quad (8)$$

$$\varphi_1(\delta)^T + \varphi_2(\delta)^T \Delta(\delta) \geq 0 \quad (9)$$

is feasible for all $\delta \in \boldsymbol{\delta}$*. Moreover, in such a case, the* L_1*-gain of the transfer from* $w_1 \rightarrow z_1$ *is bounded from above by* γ*.*

- Robust linear program
- Polynomial dependence $\rightarrow$ Handelman's Theorem (preserve linear program structure)

Examples

Example 1 - Positive time-delay system

Let us consider a positive time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-h)$$

where A is Metzler and B is nonnegative.

Linear Fractional Transformation

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw_0(t) \\ z_0(t) & = & x(t) \\ w_0(t) & = & \nabla_h(z_0)(t) \end{array}$$

where ∇_h is the constant delay operator with transfer function e^{-sh}.

Stability conditions

$$\begin{array}{rcl} \lambda^T A + \varphi^T & < & 0 \\ \lambda^T B - \varphi^T & < & 0 \end{array}$$

for some $\varphi \in \mathbb{R}^n$. Condition equivalent to

$$\lambda^T(A+B) < 0.$$

Example 2 - Positive system with parametric uncertainty (1)

Let us consider the positive uncertain system with constant parametric uncertainty $\delta \in [0,1]$:

$$\begin{array}{rcl} \dot{x}(t) & = & (A_0 + \delta A_1 + \delta^2 A_2)x(t) + (E_0 + \delta E_1 + \delta^2 E_2)w_1(t) \\ z_1(t) & = & (C_0 + \delta C_1 + \delta^2 C_2)x(t) + (F_0 + \delta F_1 + \delta^2 F_2)w_1(t) \end{array} \quad (10)$$

- 3 states, 2 inputs and 2 outputs

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_1-gain	time
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0, \varphi_1^0 + \varphi_2^0 \geq 0$	133.95	2.7844s
$\varphi_1^1\delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	133.95	3.829s
$\varphi_1^1\delta + \varphi_1^2\delta^2$	$\varphi_2^0 + \varphi_2^1\delta$	$\varphi_1^1 = -\varphi_2^0, \varphi_1^2 = -\varphi_2^1$	94.167	4.2758s

Table: L_1-gain computation of the transfer $w_1 \to z_1$ – Exact L_1-gain: 92.8358

Example 2 - Positive system with parametric uncertainty (2)

Let us consider the positive uncertain system with constant parametric uncertainty $\delta \in [0,1]$:

$$\begin{array}{rcl} \dot{x}(t) & = & (A_0 + \delta A_1 + \delta^2 A_2)x(t) + (E_0 + \delta E_1 + \delta^2 E_2)w_1(t) \\ z_1(t) & = & (C_0 + \delta C_1 + \delta^2 C_2)x(t) + (F_0 + \delta F_1 + \delta^2 F_2)w_1(t) \end{array} \quad (11)$$

- 3 states, 2 inputs and 2 outputs

$\varphi_1(\delta)$	$\varphi_2(\delta)$	constraints	computed L_∞-gain	time
φ_1^0	φ_2^0	$\varphi_1^0 \geq 0,\ \varphi_1^0 + \varphi_2^0 \geq 0$	86.195	0.68989s
$\varphi_1^1\delta$	φ_2^0	$\varphi_1^1 = -\varphi_2^0$	86.195	1.4629s
$\varphi_1^1\delta + \varphi_1^2\delta^2$	$\varphi_2^0 + \varphi_2^1\delta$	$\varphi_1^1 = -\varphi_2^0,\ \varphi_1^2 = -\varphi_2^1$	82.025	1.7509s

Table: L_∞-gain computation of the transfer $w_1 \to z_1$ – Exact L_∞-gain: 82.0249

Conclusion

Conclusion and Future Works

Conclusion

- Computing the L_1-gain of positive systems $\Leftrightarrow$ Solving a linear programming problem
- Computing of the L_∞-gain of positive systems $\Leftrightarrow$ Computing the L_1-gain of positive systems
- Robustness analysis can be done in this framework (possibly nonconservative)
- Possible improvements over the L_2-gain

Future Works

- Controller design (state-feedback, structured, static-output, with bounded coefficients): linear programming problem
- Design of dynamic output feedback ?
- Application to a real process

Thank you for your attention

Handelman's Theorem

Theorem

Let Δ be a convex polyhedra in $\mathbb{R}^N$ and a family $\mathcal{G}$ of linear functions $g_i(x) = \alpha_i^T x + \beta_i$ such that

$$\Delta = \left\{x \in \mathbb{R}^N : \; g_i(x) \geq 0\right\}.$$

Then any polynomial nonnegative over Δ can be rewritten in terms of a nonnegative linear combination of powers of the g_i's.

Example

- $p(x)$ is a polynomial of degree 2 nonnegative on the interval $[-1, 1]$
- Basis: $g_1(x) = x + 1$ and $g_2(x) = 1 - x$
- The Handelman's Theorem claims that there exist $\tau_i \geq 0$, $i = 1, \ldots, 5$ such that

$$\begin{aligned} p(x) &= \alpha_2 x^2 + \alpha_1 x + \alpha_0 \\ &= \tau_1 g_1(x) + \tau_2 g_2(x) + \tau_3 g_1(x) g_2(x) + \tau_4 g_1(x)^2 + \tau_5 g_2(x)^2 \end{aligned}$$