$\mathcal{H}_\infty$ Filtering of Uncertain LPV Systems with Time-Delays

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***Abstract*— This paper deals with the filtering of uncertain LPV systems with time-delays. First we provide a bounded real lemma for LPV time-delay systems having a more useful structure to the filtering problem than the bounded real lemma directly obtained from the parameter dependent Lyapunov-Krasovskii functional. Using the latter result we derive sufficient conditions (in terms of parametrized LMIs) for the existence of memoryless and memory $\mathcal{H}_\infty$ robust LPV filters. The effectiveness of the proposed approach is shown compared to existing ones in the literature.**

***Index Terms*— Linear parameter-varying systems; Delay systems; Filtering**

I. INTRODUCTION

The filtering problem of uncertain linear systems is related to the estimation of a linear combination of the system state and eventually some disturbances despite of model uncertainties. Although the filtering of finite dimensional linear systems has been solved in many frameworks, the case of time-delay systems is still an open problem for two main reasons: 1) The efficient LMI formulations for such problems are conservative and the actual delay margin is rarely found exactly. 2) The delay may be not measurable except for some bounds which makes the problem more difficult. Some results have been provided in [1], [2] where the descriptor model transformation of time-delay systems is considered. In [3]–[5] this approach is generalized to LPV time-delay systems. We propose in the paper an approach based on a parameter dependent Lyapunov-Krasovskii functional where we avoid model transformations/cross-terms and reducing the number of bounding procedures to the smallest number. The Lyapunov-Krasovskii functional proposed in this article is rather simple but coupled with sufficiently accurate bounding techniques, it leads to promising results.

The main contributions of the paper are the following

1) First we provide a delay-dependent stability test with $\mathcal{H}_\infty$ performances for LPV time-delay systems. This LMI has the benefit to better cope with the filtering problem.
2) Second, two solutions of the filtering problem are presented: both memory and memoryless filters are treated.
3) Finally we show the effectiveness of the approach through several examples and compare it to existing solutions in the literature.

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The paper is structured as follows: Section II introduces the problem and gives a preliminary result. Section III-B provides the main results of the paper: the solutions for the delayed-filter and the memoryless one. Section IV illustrates the approach through examples and a comparison to previous methods is discussed.

The notations are quite standard except for $A^H = A + A^T$. The operator $\times_{i=1}^{k}\mathcal{I}_i$ is the cartesian product of the sets $\mathcal{I}_i$ for all $i = 1, \ldots, k$.

II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

We consider LPV time-delay systems with polynomial parameter dependence of the form:

$$\begin{array}{rcl}\dot{x}(t) & = & (A(\rho) + \Delta A(\rho))x(t) \\ & & +(A_h(\rho) + \Delta A_h(\rho))x(t-h(t)) \\ & & +(E(\rho) + \Delta E(\rho))w(t) \\ y(t) & = & (C_y(\rho) + \Delta C_y(\rho))x(t) \\ & & +(C_{yh}(\rho) + \Delta C_{yh}(\rho))x(t-h(t)) \\ & & +(F_y(\rho) + \Delta F_y(\rho))w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t)\end{array} \quad (2)$$

where $x \in \mathbb{R}^n$, $h \in [0, h_M]$, $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ are respectively the state, the delay, the disturbances, the measurements and the controlled output. The uncertain system matrices obey the following relaxation

$$\begin{bmatrix}\Delta A(\rho) & \Delta A_h(\rho) & \Delta E(\rho) \\ \Delta C_y(\rho) & \Delta C_{yh}(\rho) & \Delta F_y(\rho)\end{bmatrix} = H(\rho)\mathbf{\Delta}_r G(\rho)$$

$$\begin{array}{rcl}H(\rho) & = & \mathrm{diag}(H_0(\rho), H_1(\rho)) \\ \mathbf{\Delta}_r & = & \mathrm{diag}(\mathbf{\Delta}, \mathbf{\Delta}) \\ G(\rho) & = & \begin{bmatrix}G_0(\rho) & G_1(\rho) & G_2(\rho) \\ G_3(\rho) & G_4(\rho) & G_5(\rho)\end{bmatrix}\end{array}$$

where $\mathbf{\Delta}$ is a constant unknown matrix such that $\mathbf{\Delta}^T\mathbf{\Delta} \preceq I_\delta$ and the H_i, G_i are known parameter dependent matrices specifying how the uncertainty $\mathbf{\Delta}$ enters the system.

The N_p parameters evolve in a compact subset U_ρ of $\mathbb{R}^{N_p}$ defined by

$$U_\rho := \times_{i=1}^{N_p}[-\underline{\rho}_i, \bar{\rho}_i]$$

where $\underline{\rho}_i$ and $\bar{\rho}_i$ denotes respectively the minimal and maximal values which may be taken by $\rho_i(t)$. Since this set is an hyper rectangle then it is a convex set. Finally, it is assumed that all the parameters can be measured in real time.

We also assume that the derivatives of the parameters are bounded and remain in a compact set of $\mathbb{R}^{N_p}$ too. It is convenient to define the set of vertices of the set of derivatives values:

$$U_\nu := \times_{i=1}^{N_p}\{\underline{\nu}_i, \bar{\nu}_i\}$$

$$\mathcal{L} = \begin{bmatrix} -X^H & P(\rho)+X^TA(\rho) & X^TA_h(\rho) & X^TE(\rho) & 0 & X^T & h_MR \\ \star & \frac{\partial P}{\partial \rho}\nu + Q - P(\rho) - R & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q - R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_w & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_z & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_MR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} < 0 \quad (1)$$

where $\underline{\nu}_i$ and $\bar{\nu}_i$ are defined by $\underline{\nu}_i \leq \dot{\rho}(t) \leq \bar{\nu}_i$ for all $t \geq 0$.

The delay $h(t)$ is assumed to satisfy the following classical assumptions

1) $h(t) \in [0, h_M]$ for all $t \geq 0$
2) $|\dot{h}| < 1$

Definition 2.1: The $\mathcal{H}_\infty$ filtering problem consists in finding a filter of the form

$$\begin{array}{rcl} \dot{x}_F(t) & = & A_F(\rho)x_F(t) + A_{Fh}(\rho)x_F(t-h(t)) \\ & & +B_F(\rho)y(t) \\ z_F(t) & = & C_F(\rho)x_F(t) + C_{Fh}(\rho)x_F(t-h(t)) \\ & & +D_F(\rho)y(t) \end{array} \quad (3)$$

such that the estimation error $e(t) := x(t) - x_F(t)$ is asymptotically stable and

$$\frac{||z_e||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < \gamma$$

where $z_e = z - z_F$. Moreover, when $A_{Fh} = 0, C_{Fh} = 0$, the filter is called memoryless.

This paper addresses the design of filters with memory when the delay is assumed to be known, measured or estimated [6]. On the other hand, when the delay is unknown, memoryless filters will be designed (i.e. $A_h = 0$, $C_h = 0$). It is clear that the former filter is difficult to implement due to the necessity of an exact knowledge of the delay value. Robustness issues with respect to a delay uncertainties may be found for instance in [7]–[10], and will be considered in another paper.

The following lemma establishes a sufficient condition to the stability of system (2).

Lemma 2.1: The LPV time-delay system (2) without uncertainties is asymptotically stable for all $h \in [0, h_M]$ and satisfies $||z||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$ if there exist a constant matrix X, a continuously differentiable bounded symmetric matrix function $P : U_\rho \to \mathbb{S}^n_{++}$, constant symmetric matrices $Q, R > 0$ and a scalar $\gamma > 0$ such that the LMI (1) holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$.

Proof: The proof is only sketched for sake of brevity. The result is based on the Lyapunov-Krasovskii functional

$$\begin{array}{rcl} V(x,\rho) & = & x(t)^TP(\rho)x(t) + \int_{t-h(t)}^{t} x(\theta)^TQx(\theta)d\theta \\ & & + \int_{-h_M}^{0}\int_{t+\theta}^{t} x(\eta)^T(h_MR)x(\eta)d\eta d\theta \end{array}$$

used along with the Jensen's inequality [11]. The matrix X is introduced using a relaxation procedure as used in [10]. Finally, the robustness part is ensured using the bounding lemma [12], [13]. ∎

The main advantage of the previous lemma comes from the fact that the data matrices are coupled with only one decision matrix (i.e. X) which allows to find a linearizing change of variable in the filtering problem, making it convex and then easily tractable using interior point algorithms [14], [15]. Moreover, this result is derived from a LMI which is well suited to the stability problem (see [16]).

Remark 2.1: In order to reduce the conservatism of the approach it is possible to consider a parameter dependent matrix $X(\rho)$.

In the remaining of the paper, the dependence on ρ is omitted for simplicity when there is no confusion.

III. Main results

In order to derive sufficient conditions of the existence of a parameter dependent filter, it is necessary to construct the extended system:

$$\begin{array}{rcl} \dot{x}_a(t) & = & \bar{\mathcal{A}}x_a(t) + \bar{\mathcal{A}}_hx_a(t-h(t)) + \bar{\mathcal{E}}w(t) \\ z_e(t) & = & \bar{\mathcal{C}}x_a(t) + \bar{\mathcal{C}}_hx_a(t-h(t)) + \bar{\mathcal{F}}w(t) \end{array} \quad (4)$$

which gathers both system and filter dynamics, that is $x_a(t) = col(x(t), e(t))$, $e(t) = x(t) - x_F(t)$,

$$\begin{array}{rcl} \bar{\mathcal{A}} & = & \begin{bmatrix} A+\Delta A & 0 \\ A+\Delta A - B_F(C_y+\Delta C_y) - A_F & A_F \end{bmatrix} \\ \bar{\mathcal{A}}_h & = & \begin{bmatrix} A_h+\Delta A_h & 0 \\ A_h+\Delta A_h - B_F(C_{yh}+\Delta C_{yh}) - A_{Fh} & A_{Fh} \end{bmatrix} \\ \bar{\mathcal{E}} & = & \begin{bmatrix} E+\Delta E \\ E+\Delta E - B_F(F_y+\Delta F_y) \end{bmatrix} \\ \bar{\mathcal{C}} & = & \begin{bmatrix} C - D_F(C_y+\Delta C_y) - C_F & C_F \end{bmatrix} \\ \bar{\mathcal{C}}_h & = & \begin{bmatrix} C_h - D_F(C_{yh}+\Delta C_{yh}) - C_{Fh} & C_{Fh} \end{bmatrix} \\ \bar{\mathcal{F}} & = & F - D_F(F_y+\Delta F_y) \end{array}$$

The goal is to find matrices $A_F, A_{Fh}, B_F, C_F, C_{Fh}D_F$ such that $\dot{x}_a(t) \to 0$ as $t \to +\infty$ when $w \equiv 0$ and $||z_e||_{\mathcal{L}_2}/||w||_{\mathcal{L}_2} < \gamma$ when $||w||_{\mathcal{L}_2} \neq 0$.

A. Existence of Robust $\mathcal{H}_\infty$ Parameter Dependent Filters

In this section, we give a certificate of existence of robust filters. It allows to compute the best $\mathcal{H}_\infty$ norm that can be achieved with the formulation presented in this paper. This 'optimal' bound is highly related to the choice of the parameter dependent matrix $P(\rho)$. Indeed, the more complex the structure of $P(\rho)$ is, the smaller is the optimal upper bound γ on $\mathcal{H}_\infty$.

Theorem 3.1: There exists an $\mathcal{H}_\infty$ robust parameter dependent filter of the form (3) for system (2) satisfying

$$
\Psi(\rho,\nu) = \begin{bmatrix}
-X^H & P + \begin{bmatrix} X_{13}^T A & 0 \\ X_{23}^T A & 0 \end{bmatrix} & \begin{bmatrix} X_{13}^T A_h & 0 \\ X_{23}^T A_h & 0 \end{bmatrix} & \begin{bmatrix} X_{13}^T E \\ X_{23}^T E \end{bmatrix} & 0 & X^T & h_M R & \begin{bmatrix} X_{13}^T H_0 & 0 \\ X_{23}^T H_0 & 0 \end{bmatrix} \\
\star & \Theta_{22} & \Theta_{23} & \Theta_{24} & \begin{bmatrix} C^T \\ 0 \end{bmatrix} & 0 & 0 & 0 \\
\star & \star & \Theta_{33} & \Theta_{34} & \begin{bmatrix} C_h^T \\ 0 \end{bmatrix} & 0 & 0 & 0 \\
\star & \star & \star & \Theta_{44} & F^T & 0 & 0 & 0 \\
\star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -P & -h_M R & 0 \\
\star & \star & \star & \star & \star & \star & -R & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\Theta
\end{bmatrix} \tag{5}
$$

Definition 2.1 if there exist a matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_3 \end{bmatrix}$, a continuously differentiable symmetric matrix $P(\rho) > 0$, constant symmetric matrices $Q, R > 0$ and scalars $\gamma, \varepsilon > 0$ such that LMIs

$$\mathcal{N}_U^T \Psi(\rho,\nu) \mathcal{N}_U < 0 \tag{6}$$

$$\mathcal{N}_V^T \Psi(\rho,\nu) \mathcal{N}_V < 0 \tag{7}$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ with Ψ is defined in (5) and where

$$
\begin{aligned}
X_{13} &= X_1 + X_3 \\
X_{23} &= X_2 + X_3 \\
\Theta_{22} &= \frac{\partial P}{\partial \rho}\nu - P + Q - R + \varepsilon \begin{bmatrix} G_0^T G_0 + G_3^T G_3 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{23} &= R + \varepsilon \begin{bmatrix} G_0^T G_1 + G_3^T G_4 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{24} &= \varepsilon \begin{bmatrix} G_0^T G_2 + G_3^T G_5 \\ 0 \end{bmatrix} \\
\Theta_{33} &= -(1-\mu)Q - R + \varepsilon \begin{bmatrix} G_1^T G_1 + G_4^T G_4 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{34} &= \varepsilon \begin{bmatrix} G_1^T G_2 + G_4^T G_5 \\ 0 \end{bmatrix} \\
\Theta_{44} &= -\gamma I_m + \varepsilon (G_2^T G_2 + G_5^T G_5) \\
\mathcal{N}_U &= Ker[U] \\
\mathcal{N}_V &= Ker[V^T]
\end{aligned}
$$

$$
U = \left[\begin{array}{cc}
-I & 0 \\ -I & 0 \\ \hline
0 & 0 \\ 0 & 0 \\ \hline
0 & 0 \\ 0 & 0 \\ \hline
0 & 0 \\ \hline
0 & -I \\ \hline
0 & 0 \\ 0 & 0 \\ \hline
0 & 0 \\ 0 & 0 \\ \hline
0 & 0 \\ 0 & 0
\end{array}\right]
\qquad
V = \left[\begin{array}{ccc}
0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline
I & 0 & C_y^T \\ -I & 0 & 0 \\ \hline
0 & I & C_{yh}^T \\ 0 & -I & 0 \\ \hline
0 & 0 & F_y^T \\ \hline
0 & 0 & 0 \\ \hline
0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline
0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline
0 & 0 & 0 \\ 0 & 0 & H_1^T
\end{array}\right] \tag{8}
$$

Proof: The proof is based on the substitution of the the extended system (4) into LMI (1) with $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_3 \end{bmatrix}$. Then the projection lemma [17] is applied in order to eliminate the filter matrix variables. ■

This theorem has no vocation to synthesize a filter but gives only an indication on the best $\mathcal{H}_\infty$ norm that can be satisfied by the transfer from the disturbances w to the estimation error vector z_e.

Remark 3.1: To deal with memoryless filters, just remove the second column of $\mathcal{F}(\rho)$ and V.

It is possible to construct a filter from this existence condition using two different ways: either an implicit construction using a semidefinite programming problem or an explicit one through an algebraic construction.

1) Semidefinite Programming Reconstruction: After solving LMIs (6) and (7) for $P(\rho), R, Q, \gamma, \varepsilon, X$ the inequality

$$\Psi(\rho,\nu) + U^T \mathcal{F}(\rho) V(\rho) + V(\rho)^T \mathcal{F}(\rho)^T U < 0 \tag{9}$$

is linear in $\mathcal{F}(\rho)$ where $\mathcal{F}(\rho) = \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix}$.

The previous problem is semi-infinite infinite dimensional LMI and hence after decomposing the matrix function $\mathcal{F}(\rho)$ on a specific basis ([18]) it is possible to solve the problem using interior points algorithms ([14], [15]). To relax the semi-infinite part (the dependence on the parameters), it is possible to use several approaches [18], [19], [20], [21].

Finally the filter matrices $\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \end{bmatrix}$ can be obtained using the expression

$$X_3^{-T} \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \end{bmatrix}$$

2) Algebraic construction: The algebraic construction can only be used when the matrix P is chosen constant (i.e. $P(\rho) = P_0$). This is due to the fact that if P is varying then the resulting filter would depend on the parameter derivative $\dot{\rho}$ whose knowledge rarely occurs in practice.

The formula is obtained through algebraic computations from the inequality (9); this is detailed in the papers [22], [23].

Algorithm 3.1:

1) Find $\lambda > 0$ such that $\Phi(\rho) := (\lambda V(\rho)^T V(\rho) - \Psi(\rho))^{-1} > 0$
2) Compute

$$\mathcal{F}(\rho) = -\lambda V(\rho) \Phi(\rho) U \left[U^T \Phi(\rho) U \right]^{-1}$$

where U, V are defined in (8) and $\Psi(\rho) = \Psi(\rho,\nu)_{|\nu=0}$ which is defined in (5).

3) $\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \end{bmatrix} = X_3^{-T} \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \end{bmatrix}$

The main advantage of this formulation is that the filter is directly obtained from the structure of the system and the matrix P. There is no need to express $\mathcal{F}(\rho)$ on a particular basis but the conservatism may be high since we do not restrict the bounds on the parameters variation rate.

B. Direct Computation of Robust $\mathcal{H}_\infty$ Parameter Dependent Filters

The following theorem is an alternative to the implicit reconstruction of the filter. In the following theorem the filter is directly computed using a global SDP problem.

Theorem 3.2: There exists an $\mathcal{H}_\infty$ robust parameter dependent filter of the form (3) for system (2) satisfying the objectives if there exist a matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_3 \end{bmatrix}$, a continuously differentiable symmetric matrix $P(\rho)$, constant symmetric matrices Q, R, matrix functions $\tilde{A}_F(\rho)$, $\tilde{A}_{Fh}(\rho)$, $\tilde{B}_F(\rho)$, $C_F(\rho)$, $C_{Fh}(\rho)$, $D_F(\rho)$ and scalars $\gamma > 0, \varepsilon > 0$ such that LMI

$$\begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix} < 0 \tag{10}$$

holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$ where $M_{11} = \begin{bmatrix} X^H & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 \\ \star & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25} \\ \star & \star & \Theta_{33} & \Theta_{34} & \Theta_{35} \\ \star & \star & \star & \Theta_{44} & \Theta_{45} \\ \star & \star & \star & \star & -\gamma I_q \end{bmatrix}$, $M_{12} = \begin{bmatrix} -X^T & h_M R & \Theta_{18} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Theta_{58} \end{bmatrix}$, $M_{22} = \begin{bmatrix} -P(\rho) & -h_M R & 0 \\ \star & -R & 0 \\ \star & \star & -\varepsilon I \end{bmatrix}$, $X_{13} = X_1 + X_3$, $X_{23} = X_2 + X_3$,

$$\begin{aligned}
\Theta_{12} &= P + \begin{bmatrix} X_{13}^T A - \tilde{B}_F C_y - \tilde{A}_F & \tilde{A}_F \\ X_{23}^T A - \tilde{B}_F C_y - \tilde{A}_F & \tilde{A}_F \end{bmatrix} \\
\Theta_{13} &= \begin{bmatrix} X_{13}^T A_h - \tilde{B}_F C_{yh} - \tilde{A}_{Fh} & \tilde{A}_{Fh} \\ X_{23}^T A_h - \tilde{B}_F C_{yh} - \tilde{A}_{Fh} & \tilde{A}_{Fh} \end{bmatrix} \\
\Theta_{14} &= \begin{bmatrix} X_{13}^T E - \tilde{B}_F F_y \\ X_{23}^T E - \tilde{B}_F F_y \end{bmatrix} \\
\Theta_{18} &= \begin{bmatrix} X_{13}^T H_0 & -\tilde{B}_F H_1 \\ X_{23}^T H_0 & -\tilde{B}_F H_1 \end{bmatrix} \\
\Theta_{22} &= \frac{\partial P}{\partial \rho}\nu - P + Q - R + \varepsilon \begin{bmatrix} G_0^T G_0 + G_3^T G_3 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{23} &= R + \varepsilon \begin{bmatrix} G_0^T G_1 + G_3^T G_4 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{24} &= \varepsilon \begin{bmatrix} G_0^T G_2 + G_3^T G_5 \\ 0 \end{bmatrix} \\
\Theta_{25} &= \begin{bmatrix} C - D_F C_y - C_F & C_F \end{bmatrix}^T \\
\Theta_{33} &= -(1-\mu)Q - R + \varepsilon \begin{bmatrix} G_1^T G_1 + G_4^T G_4 & 0 \\ 0 & 0 \end{bmatrix} \\
\Theta_{34} &= \varepsilon \begin{bmatrix} G_1^T G_2 + G_4^T G_5 \\ 0 \end{bmatrix} \\
\Theta_{35} &= \begin{bmatrix} C_h - D_F C_{yh} - C_{Fh} & C_{Fh} \end{bmatrix}^T \\
\Theta_{44} &= -\gamma I_m + \varepsilon (G_2^T G_2 + G_5^T G_5) \\
\Theta_{45} &= (F - D_F F_y)^T \\
\Theta_{58} &= \begin{bmatrix} 0 & -D_F H_1 \end{bmatrix}
\end{aligned}$$

Moreover, the whole filter matrices can be constructed using the relations $A_F(\rho) = X_3^{-T}\tilde{A}_F(\rho)$, $A_{Fh}(\rho) = X_3^{-T}\tilde{A}_{Fh}(\rho)$ and $B_F(\rho) = X_3^{-T}\tilde{B}_F(\rho)$

Proof: The proof follows the same lines as the proof of Theorem 3.1 but keeps the filter matrices in the LMIs instead of eliminating them. ■

For practical reasons we propose a memoryless version of the filter since, in many cases, the delay cannot be known in real-time.

Corollary 3.1: There exists a memoryless $\mathcal{H}_\infty$ robust parameter dependent filter of the form (3) (with $A_F = 0$ and $C_{Fh} = 0$) satisfying the objectives if there exist a matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_3 \end{bmatrix}$, a continuously differentiable symmetric matrix $P(\rho)$, constant symmetric matrices Q, R, matrix functions $\tilde{A}_F(\rho)$, $\tilde{B}_F(\rho)$, $C_F(\rho)$, $D_F(\rho)$ and scalars $\gamma > 0, \varepsilon > 0$ such that LMI (10) holds with $\tilde{A}_{Fh}(\rho)$ and $\tilde{C}_{Fh}(\rho)$ are both set to 0 for all $\rho \in U_\rho$ and $\nu \in U_\nu$.

Proof: Just set to 0 the matrices A_{Fh} and C_{Fh} of the filter in LMI (10). ■

IV. EXAMPLES

A. Example 1

We consider here the LTI time-delay system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x_h \\
&\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\
z &= \begin{bmatrix} 1 & 2 \end{bmatrix} x \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{aligned} \tag{11}$$

Our method is compared to the result of [2] for different values of μ (in the memoryless and single delay case) and we obtain results in Table I, for the $\mathcal{H}_\infty$ attenuation level γ.

μ	0	0.4	0.8
[2]	1.4086	1.8311	15.8414
Theorem 3.2	0.06484	0.10651	0.48661

TABLE I

ACHIEVED $\mathcal{H}_\infty$ PERFORMANCE FOR DIFFERENT VALUES OF μ FOR $h_M = 1$.

This shows the proposed approach, for the synthesis of memoryless filters, clearly outperforms the approach in [2] for LTI time-delay systems.

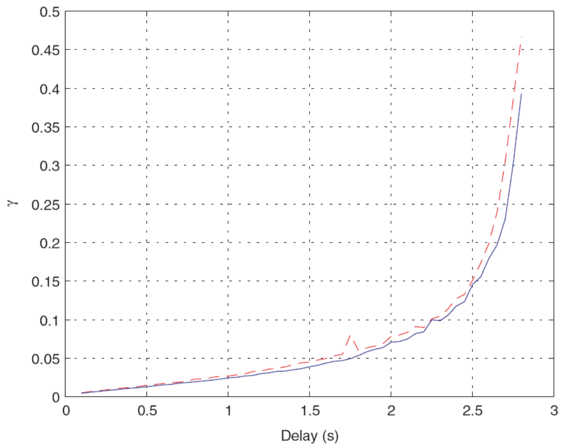

Fig. 1. Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain) in [5]

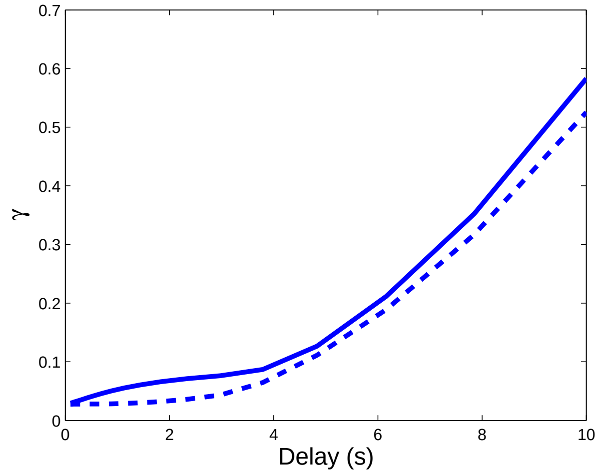

Fig. 2. Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain) from Theorem 3.2

B. Example 2

Consider the following system [5]:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h \\ &\quad + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w \\ z &= \begin{bmatrix} 0.3 & 1.5 \\ -0.45 & 0.75 \end{bmatrix} x + \begin{bmatrix} 0.5\rho & -0.5 \end{bmatrix} w \\ y &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1+0.1\rho \end{bmatrix} w \end{aligned} \quad (12)$$

where $\rho(t) = sin(t) \in [-1, 1]$ and $\dot{\rho}(t) \in [-1, 1]$.

All the parameter dependent matrices are expressed onto a basis formed by the functions

$$G_0(\rho) = 1 \qquad G_1(\rho) = \rho \quad (13)$$

We use theorem 3.2 with an uniform gridding of 11 points over the whole parameter space and the results are verified over a thinner grid.

Results of [5] are depicted in Figure 1. In figure 2, the evolution of the worst case performance for the delayed filter and the memoryless one is shown. As a first analysis, the delayed filter gives better performance than the memoryless one which seems obvious since the information on the delay is used in the delayed filter. As a comparison with the results in [5], our results are less conservative and then improve the existing ones (see result of [5] in figure 1 and proposed results in figure 2). It is possible to see that for small delay values both solutions leads to very similar results. The main difference appears for larger delay values for which the worst case disturbance gain is drastically different.

Figure 3 shows the evolution of the error $z(t) - z_F(t)$ for a delay $h = 3$ and a step disturbance of amplitude 20. We can easily see that the delayed filter gives better results than the memoryless one.

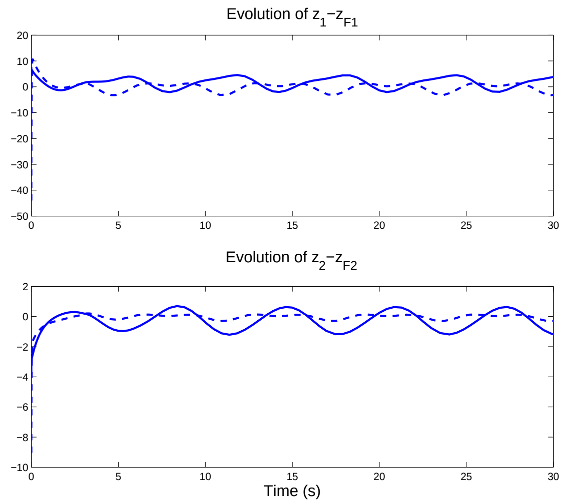

Fig. 3. Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

Consider now system (12) with matrices

$$\begin{aligned} &H_0 = H_1 = 0.1I_2 \qquad G_0 = G_1 = G_3 = G_4 = I_2 \\ &G_2 = G_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \quad (14)$$

The evolution of the worst-case performance $\mathcal{L}_2$-gain is depicted in figure 4.

Figure 5 shows the evolution of the error $z(t) - z_F(t)$ for a delay $h = 4.5$, $\Delta(t) = sin(10t)I_2$ and a step disturbance of amplitude 20. The delayed filter achieves an $\mathcal{L}_2$ performance gain of $\gamma_{del} = 0.59$ and the memoryless of $\gamma_{ml} = 0.78$.

V. Conclusion

A new approach for the filtering of LPV time-delay systems has been provided. It is based on a parameter dependent Lyapunov-Krasovskii functional of a simple form but suffices to give interesting results while used with a particular

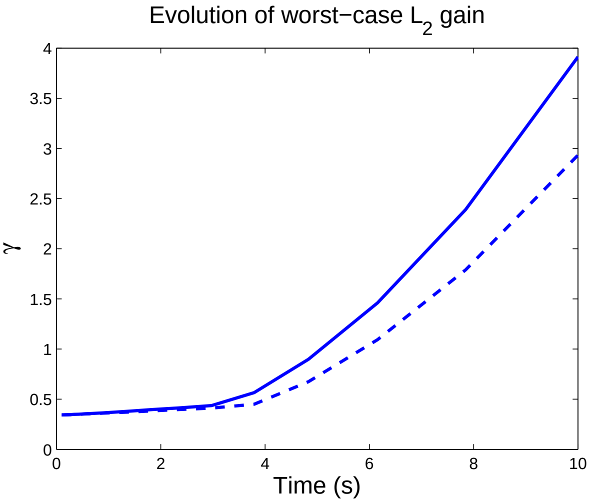

Fig. 4. Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain) from Theorem 3.2

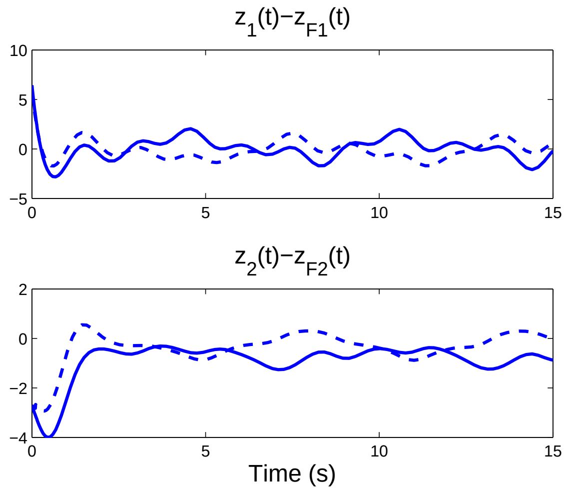

Fig. 5. Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

transformation of the resulting LMIs. This transformation allows to modify the LMIs structure and makes them able to be used for filter synthesis. Two results have been provided, the first one is simply an existence lemma which does not depend on the filter structure w.r.t the parameters. It allows to give a certificate of the best $\mathcal{H}_\infty$ norm that can be achieved by this method. The second result is a direct computation of the filter with a given structure. The efficiency of the approach has been shown through several examples.

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