A LFT/$\mathcal{H}_\infty$ state feedback design for Linear Parameter Varying Time Delay Systems

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***Abstract*— This paper considers $\mathcal{H}_\infty$ control of LPV time delay systems in LFT form. We first introduce a new model transformation for systems with time varying delay which allows us to propose a systematic method to design gain-scheduled state feedback in the delay dependent framework. Further, we extend this method to elaborate a new type of controller for TDS whenever the delay is measured in real time and explicitly used in the controller as a parameter. These results are presented with single delayed systems but can be easily generalized to multiple delayed systems.**

***Index Terms*— Time delay systems, linear parameter varying systems, robust control**

I. Introduction

Since several years constant time-delay systems (TDS) have been widely studied (See [7], [13], [21], [3], [5], [30], [15] and references therein). More recently, time varying delays, appearing for example in communication networks, have suggested more and more interest (See [6], [9], [22], [27],[8] and references therein). Indeed, lags in communication channels may destabilize systems or deteriorate performances. Two types of stability criteria are often used: delay independent and delay dependent. The latter treats only delays which belongs to a compact set and is, in general, less conservative and more precise than the delay independent criteria. Indeed, in practice, the delays are bounded and thus, delay dependent stability is preferred.

In the last two decades, linear parameter varying system (LPV systems) have been studied and several stability analysis and control synthesis methods exist (See [16], [1], [2], [17], [10], [25], [19], [11], [26], [18], [20], [14] and references therein). Three principal methods have been developed: the polytopic approach, the gridding method and the LFT approach. The first one uses the parameter values at each vertex of the polytope, but needs an online convex combination of an exponential number of controllers at each instant, which may be actually computationally expensive (the design may also require a large number of LMI constraints). The great benefits of this approach is its the synthesize simplicity and a good conservatism reduction using for example slack variables. The second one is the gridding method using infinite dimensional LMIs discretized over the parameter space. This leads to simple controllers to implement but the type and the number of basis functions, the number and the space between discretization points are not well defined. The last is the LFT approach which consists in pulling out the varying parameters, as done usually with uncertainties in robust control. According to the type of scalings (multipliers) used, this approach may leads to a small number to a exponential number of LMIs where decision matrices do not depend on parameters (which is conservative). The conservatism reductionis performed using scalings in the small-gain theorem argument. Moreover, an unique and easily implementable controller is obtained which is self smoothly scheduled by the parameters using elementary arithmetical operations. Each method has its on benefits and disadvantages. In this paper we have only interested to develop a LFT approach.

We consider in this paper LPV time-delay systems of the form:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\theta)x(t)+A_h(\theta)x_h(t)\dots\\ & & +B_u(\theta)u(t)+B_w(\theta)w(t)\\ z(t) & = & C_z(\theta)x(t)+C_{zh}(\theta)x_h(t)\dots\\ & & +D_{zu}(\theta)u(t)+D_{zw}(\theta)w(t)\end{array}\tag{1}$$

where $x_h(t)=x(t-h(t))$, h is the time varying time delay, x is the state, u the control inputs, w the exogenous inputs and z the controlled outputs. $\theta=col(\theta_i)$ is a possibly time varying parameter vector whose components belong to the set

$$\mathscr{P}:=\{\theta_i(t)\in[-1,1],\ i=1,\dots,N\}\tag{2}$$

Note that the parameters time derivative are unconstrained and since, in practice, it may lead to additional conservatism. Nevertheless, the set $\mathscr{P}$ may be extended to a larger set containing static/dynamic nonlinearities, memoryless operators... whose derivatives are difficult (even impossible) to obtain. It is proposed in [11] a method tackling parameter derivatives bur is difficult to adapt in the proposed synthesis. If only the bounds are known, a robust synthesis should be done with respect to these bounds but this will not reduces conservatism. If their derivatives are known at each instant, the controller may be scheduled by the parameters and their derivatives. In practice, it is possible to compute the derivatives simply using a second order low pass filter.

The parameters matrix is diagonal

$$\Theta(t):=\bigoplus_i\theta_i(t)I_{p_i}\tag{3}$$

where $\theta_i(t)$ are the time varying parameters and p_i is the number of occurrences of the i^{th} parameter.

LPV time-delay systems have been studied in [31], [24], [28], [29], [23] but using polytopic approach or gridding

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methods only. However, the LFT approach has not been really extended to these kinds of systems.

This paper brings a new approach to study delay-dependent stability and control of LTI/LPV time-delay systems with interval time-varying delay (ie. $0 < h_m < h(t) < h_M < +\infty$), in an unified framework of robust control theory for LPV systems.

The contributions of the paper are the following. First a new model transformation for TDS with interval time varying delays is introduced which transforms a time delay system into an uncertain LPV system. Hence a classical stability test is sufficient to conclude.

Second, we express sufficient conditions to the existence and computation feasibility of gain-scheduled state-feedback memoryless controllers, which will depend on the delay bounds and the parameters. The controllers are represented in a LFT form and are not unique. This non-uniqueness allows to the designer to add supplementary constraints to the synthesis problem such as robust pole placement, resilience…(despite of render it, in some case, unfeasible).

Third, we exploit the fact that our approach permits to transform a (LPV) time-delay system into an uncertain LPV system where the delay does not act anymore as an operator but as a bounded time-varying parameter. This allows to elaborate a new kind of controllers. The same methodology is applied to synthesize delay-scheduled controllers with approximate knowledge of the delay value. The measurement error on the delay value is here taken into account to ensure the robustness of the closed-loop system stability.

The advantage of the proposed methodology relies on the fact that we propose for the first time an unique LFT formulation to design different types of controllers for different classes of systems.

The paper is structured as follows. Section II recall useful lemmas and gain-scheduled state feedback existence theorem inspired from [1]. In section III we provide delay dependent stability and stabilization of LPV time delay systems via the use of a particular model transformation. In section IV we present a new LPV based control method for time-delay systems when delays are approximatively known and which can easily be extended to LPV time-delayed systems.

We denote by A^T the transpose of the matrix A, $Ker(A)$ is a basis of the null space of the operator A, $A > 0$ ($A < 0$) means that A is positive definite (negative definite), $\oplus$ is the direct sum of matrices, $\dot{x}(t)$ is the time derivative of the signal $x(t)$, $\mathcal{L}_2(\mathbb{R}^+,\mathbb{R})$ ($\mathcal{L}_2([-h_{max},+\infty],\mathbb{R})$) is the space of square integrable signals which maps $\mathbb{R}^+$ to $\mathbb{R}$ (from $[-h_{max},\mathbb{R}^+]$ to $\mathbb{R}$). $\mathcal{V}_{i=1}^n[A_i]$ denotes the matrix $\begin{bmatrix} A_1^T & \dots & A_n^T \end{bmatrix}^T$ (subscript and superscript are sometimes omitted when there are useless). $\star$ denotes the transpose of the symmetric term in symmetric matrices.

II. Background

Let us introduce the well-known Scaled Bounded real Lemma (SBRL):

Proposition 2.1: Scaled Bounded Real Lemma

Consider the system

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + \tilde{B}w(t) + \tilde{B}_u u(t) \\ z(t) & = & \tilde{C}x(t) + \tilde{D}w(t) + \tilde{D}_u u(t) \end{array} \tag{4}$$

where x is the state, w the exogenous inputs, u the control inputs and z the controlled outputs.

Let the set of scalings associated to the parameters structure Θ be defined as

$$L_\Theta := \left\{ L : \Theta L = L\Theta,\ L = L^T > 0 \right\} \tag{5}$$

Note that this set is convex and so this constraint can be merged with LMI ones. The set of scalings of the repeated structure $\Theta(t) \oplus \Theta(t)$ is defined as

$$L_{\Theta\oplus\Theta} := \left\{ \begin{bmatrix} L_1 & L_2 \\ \star & L_3 \end{bmatrix} > 0,\ L_1, L_3 \in L_\Theta, L_2\Theta = \Theta L_2 \right\}$$

More details about this type of scalings can be found in [1].

If there exist matrices $X = X^T > 0,\ L \in L_{\Theta\oplus\Theta}$ and a scalar $\gamma > 0$ such that

$$\begin{bmatrix} \tilde{A}^T X + X\tilde{A} & X\tilde{B} & \tilde{C}^T \\ \star & -L_w & \tilde{D}^T \\ \star & \star & -L_z^{-1} \end{bmatrix} < 0 \tag{6}$$

with $L_w = L \oplus \gamma I_w$,$L_z = L \oplus \gamma I_z$, then the system (4) is asymptotically stable and the induced $\mathcal{L}_2$ norm of the channel $[L^{1/2} \oplus 0]\,[w(t) \to z(t)]\,[L^{-1/2} \oplus 0]$ is less than 1 and $[0 \oplus I_w]\,[w(t) \to z(t)]\,[0 \oplus I_z]$ is less than γ.

More details may be found in [1].

■

The L matrix scales the parameter dependent channels (system and controller one) which reduces the conservatism of the small gain theorem despite of render this inequality non convex. However, this apparent untractable problem can be easily exactly relaxed without inducing some conservatism using the projection lemma (See [1]). The γ term acts on the performance channel which needs to be optimized.

Let us consider now the LPV system, represented as in Figure 1.

$$\begin{array}{rcl} \dot{x}(t) & = & \bar{A}x(t) + \bar{B}w(t) + \bar{B}_u u(t) \\ z(t) & = & \bar{C}x(t) + \bar{D}w(t) + \bar{D}_u u(t) \end{array} \tag{7}$$

with matrices

$$\begin{array}{rcl} \bar{B} & = & \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix} \\ \bar{C} & = & \begin{bmatrix} C_0^T & C_1^T & C_2^T \end{bmatrix}^T \\ \bar{D} & = & \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{01} & D_{11} & D_{12} \\ D_{20} & D_{21} & D_{22} \end{bmatrix} \\ \bar{D}_u & = & \begin{bmatrix} D_{0u}^T & D_{1u}^T & D_{2u}^T \end{bmatrix}^T \\ z & = & \begin{bmatrix} z_0^T & z_1^T & z_2^T \end{bmatrix}^T \\ w & = & \begin{bmatrix} w_0^T & w_1^T & w_2^T \end{bmatrix}^T \end{array} \qquad \begin{array}{rcl} w_0(t) & = & \Theta(t)z_0(t) \\ w_1(t) & = & \Delta(z_1(t)) \end{array} \tag{8}$$

where x is the state, u the control input, z_0/w_0 the LPV channel, z_1/w_1 the uncertain channel and z_2/w_2 the performance channel which has to be optimized (see figure 1).

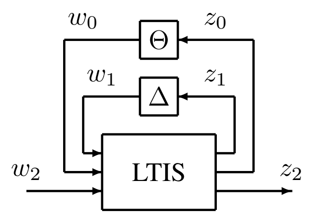

Fig. 1. LPV uncertain system

The aim of this paper is to design a gain-scheduled state feedback controllers (GSSF) of the form (9) (see figure 2)

$$\begin{array}{rcl} u(t) & = & K_{00}x(t) + K_{01}w_c(t) \\ z_c(t) & = & K_{10}x(t) + K_{11}w_c(t) \\ w_c(t) & = & \Theta(t)z_c(t) \end{array} \tag{9}$$

where u is the control input, x the system state, z_c and w_c represent the controller parameter dependent signals, such that the closed-loop system is asymptotically stable and the $\mathcal{H}_\infty$ norm of the channel $w_2 \to z_2$ is less than γ in delay-dependent and delay-scheduled framework.

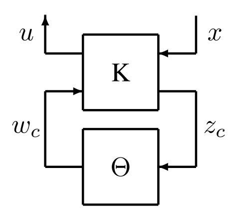

Fig. 2. LPV state feedback

We present here a LPV state feedback existence theorem inspired from [1].

Theorem 2.1: LPV state-feedback existence theorem

Consider the LPV system of the form (7)-(8) with $\bar{C}_y = I$ and $\bar{D}_{yw} = 0$, then if there exist some positive definite matrices X, L_3, J_3 and a positive scalar γ such that these LMIs hold

$$\mathcal{K}_i^T \left(\Lambda_i + T_i^T M_i T_i\right) \mathcal{K}_i < 0, \text{ for } i = 1, 2 \tag{10}$$

$$\begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0 \qquad L_3, J_3 \in L_\Theta \tag{11}$$

with

$$\begin{array}{ll} \Lambda_1 = -L_3 \oplus -I_1 \oplus -\gamma I_{w_2} & T_1 = \bar{D} \\ M_1 = L_3 \oplus I_1 \oplus \gamma^{-1} I_{z_2} & \mathcal{K}_1 = I \\ \Lambda_2 = \begin{bmatrix} Y\bar{A}^T + \bar{A}Y & Y\bar{C}^T \\ \star & -V_J \end{bmatrix} & V_J = J_3 \oplus I_2 \oplus \gamma I_{z_2} \\ T_2 = \begin{bmatrix} \bar{B}^T & \bar{D}^T \end{bmatrix} & M_2 = J_3 \oplus I_1 \oplus \gamma^{-1} I_{w_2} \\ \mathcal{K}_2 = Ker\begin{bmatrix} \bar{B}_u^T & \bar{D}_u^T \end{bmatrix} & \end{array}$$

then there exists a GSSF of the form (9) which stabilizes (7)-(8) with an achieved $\mathcal{H}_\infty$ performance index lower than γ. The controller matrices (9) can be found by solving the following LMI

$$\Psi_{sf} + P_{sf}^T K Q_{sf} + P_{sf} K^T Q_{sf}^T < 0 \tag{12}$$

where $K = \begin{bmatrix} K_{00} & K_{10} \\ K_{10} & K_{11} \end{bmatrix}$ and Ψ_{sf} is the matrix (6) with matrices

$$\begin{array}{rcl} \tilde{A} & = & \bar{A} \quad \tilde{B} = \begin{bmatrix} 0 \\ \bar{B}^T \end{bmatrix}^T \quad \tilde{C} = \begin{bmatrix} 0 \\ \bar{C} \end{bmatrix} \quad \tilde{D} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{D} \end{bmatrix} \\ L_w & = & \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix} \oplus I_1 \oplus \gamma I_{w_2} \\ L_z^{-1} & = & \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}^{-1} \oplus I_1 \oplus \gamma I_{z_2} \\ P_{sf} & = & \left[\begin{array}{cc} 0 & Y^{-1}\bar{B}_u \\ \hline 0 & 0 \\ \hline 0 & I \\ D_u & 0 \end{array}\right]^T, \ Q_{sf} = \left[\begin{array}{c|cc|c} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right] \end{array}$$

Proof: A sketch of a proof and the way to find the whole matrix L are discussed in appendix A. ■

Note that several controllers may satisfy the LMIs (12), therefore it may be useful consider other specifications robust pole placement, stable controller, small coefficients...

III. LPV CONTROLLERS SYNTHESIS - DELAY DEPENDENT FRAMEWORK

In all what follows we will consider TDS and provide some transformations such that the TDS is rewritten under the LFT form (7)-(8), which allows to use theorem 2.1.

A. A new model transformation

Consider a linear time delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x_h(t) + B_1 w_1(t) \\ z_1(t) & = & Cx(t) + C_h x_h(t) + D_{11} w_1(t) \end{array} \tag{13}$$

where x is the state, $x_h = x(t - h(t))$, w_1 the exogenous inputs and z_1 the controlled outputs. Let $\mathcal{D}_h(\cdot)$ be the operator $\mathcal{D}_h := \nu(t) \to \dfrac{h_m}{h_M} \displaystyle\int_{t-h(t)}^{t} h(\eta)^{-1}\nu(\eta)d\eta$ where $0 < h_m \leq h(t) \leq h_M < +\infty$. One can check that $\|\mathcal{D}_h\|_\infty \leq 1$ (see appendix B).

Then the system (13) may be rewritten as the following uncertain linear parameter varying where $h(t)$ is viewed as a parameter. Indeed $x_h = x - w_0$ and $z_0 = \alpha(t)\dot{x}$ then replace them into the system (13) leads to

$$\begin{array}{rcl} \dot{x} & = & (A + A_h)x(t) - A_h w_0(t) + B_1 w(t) \\ z_0(t) & = & (C + C_h)x(t) - C_h w_0(t) + D_{11} w_1(t) \\ z_1(t) & = & \alpha(t)(A + A_h)x(t) - \alpha(t)A_h w_0(t) \ldots \\ & & +\alpha(t)B_1 w_1(t) \\ w_0(t) & = & \mathcal{D}_h(z_0(t)) = x(t) - x_h(t) \\ \alpha(t) & = & h(t)\dfrac{h_M}{h_m} \end{array} \tag{14}$$

Hence, we can see that the stability of the system (14) implies the stability of system (13) because this transformation induces additional dynamics (the initial value is now defined on $[-2h_M, 0]$). We may also analyze the stability of these additional dynamics (see [8] for details). An advantage of this transformation comes from the fact that the delay bounds appear explicitly in the expression of the system and allow us to establish a delay-dependent stability test. This transformation permits to transform the time-delay system stability analysis problem into a LPV uncertain system stability analysis. A similar result is provided in [13] for constant time-delay. That means that we could use classical tools from robust analysis and control as theorem 2.1.

Note that the operator $\mathcal{D}_h(\cdot)$ is treated as an uncertainty which is less conservative than a delay independent criteria where the whole delay is considered as uncertainties. A similar model transformation has been performed for constant time delays in [12].

B. $\mathcal{L}_2$-$\mathcal{L}_2$ stability Analysis

Theorem 3.1:

Consider the transformed time-delayed system (14), if there exist symmetric positive definite matrices P, Q and a scalar $\gamma > 0$ such that the two LMIs hold

$$\begin{bmatrix} \bar{A}^T X + X\bar{A} & -XA_h & XB_1 & \alpha_i \bar{A}^T Q & \bar{C}^T \\ \star & -Q & 0 & -\alpha_i A_h^T Q & -C_h^T \\ \star & \star & -\gamma I_{w_1} & \alpha_i B_1^T Q & D_{11}^T \\ \star & \star & \star & -Q & 0 \\ \star & \star & \star & \star & -\gamma I_{z_1} \end{bmatrix} < 0 \tag{15}$$

for $i = 1, 2$ with $\alpha_1 = h_M$, $\alpha_2 = \dfrac{h_M^2}{h_m}$, $\bar{A} = A + A_h$ and $\bar{C} = C + C_h$, then the system (14) is asymptotically stable and hence time-delayed system (13) is asymptotically stable for all delays belonging to $[h_m, h_M]$ and the channel $w_1(t) \to z_1(t)$ has an induced $\mathcal{L}_2$-norm lower than γ.

Proof: The first step is to inject the matrices of the uncertain LPV system (14) into the scaled bounded real lemma LMI (6), with $L_w = L \oplus \gamma I_{w_1}$, $L_z = L \oplus \gamma I_{z_1}$. Then we perform a congruence transformation with the non-singular matrix $I \oplus I \oplus I \oplus L \oplus I$. That leads to a semi-infinite LMI written $F_1 + h(t)F_2 < 0$ where F_1, F_2 are respectively the LMI constant part and the parameter (the delay) dependent part. As the relation in $h(t)$ is affine then via a convex argument we can state that if we check the feasibility of the LMI at the vertices of the convex set where $h(t)$ evolves, then the parameter dependent LMI will be negative definite for every $h(t)$ in the convex set. Hence,it remains two LMIs only for h_m and h_M which corresponds to the LMI (15). ∎

Remark 3.1: The above theorem permits to conclude on the system stability only onto the separated delay interval $\{0\} \cup [h_m, h_M]$. If the LMI (15) is satisfied then that means that the upper-left term is negative definite and hence that the system without delay is asymptotically stable.

In the remaining parts of this section, we consider the following LPV time-delay system (see figure 3-a)

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + A_h x_h + B_0 w_0 + B_2 w_2 + B_u u \\ z_0 & = & C_0 x + D_{00} w_0 + D_{02} w_2 + D_{0u} \\ z_2 & = & C_2 x + D_{20} w_0 + D_{22} w_2 + D_{2u} u \end{array} \right. \tag{16}$$

where x is the system state, x_h the delayed state, u the control input, z_0/w_0 the LPV channel, z_2/w_2 the performance channel with the exogenous inputs w_2 and the controlled outputs z_2.

Using the model transformation presented in III-A, the above model is transformed into a LPV uncertain (memory-less) system (fig. 3-b) where Δ is the operator $\mathcal{D}_h(\cdot)$

$$\left\{ \begin{array}{rcl} \dot{x} & = & \bar{A}x + \bar{B}w + \bar{B}_u u \\ z & = & \bar{C}x + \bar{D}w + \bar{D}_u u \end{array} \right. \tag{17}$$

with $w = \begin{bmatrix} w_0^T & w_1^T & w_2^T \end{bmatrix}^T$ and $z = \begin{bmatrix} z_0^T & z_1^T & z_2^T \end{bmatrix}^T$ where we add the uncertain channel z_1/w_1.

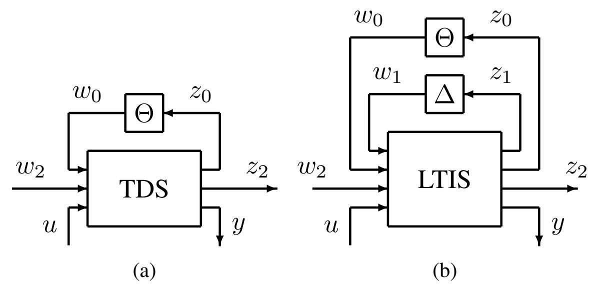

Fig. 3. (a) LPV time delay system (b) LPV uncertain system

We consider now systems of the form (17) with matrices

$$\bar{A} = A + A_h,\ \bar{D}_u^i = \begin{bmatrix} D_{0u}^T & \alpha_i \bar{B}_u^{\ T} & D_{2u}^T \end{bmatrix}^T$$

$$\bar{B} = \begin{bmatrix} B_0 & -A_h & B_2 \end{bmatrix},\ \bar{C}_y = I$$

$$\bar{C}_i = \begin{bmatrix} (C_0 + C_{0h})^T & \alpha_i \bar{A}^T & (C_2 + C_{2h})^T \end{bmatrix}^T$$

$$\bar{D}_i = \begin{bmatrix} -C_{0h} & D_{01} & D_{02} \\ -\alpha_i C_{0h} & \alpha_i B_1 & \alpha_i B_2 \\ -C_{2h} & D_{21} & D_{22} \end{bmatrix},\ \bar{D}_{yw} = \begin{bmatrix} 0 & | & 0 & | & 0 \end{bmatrix}$$

The above system is represented in a similar representation than system (7).

C. Delay dependent LPV state feedback result

We propose in this part a control law of the form $u(t) = K(\theta(t))x(t)$ in a delay dependent framework. The delay is assumed to belong to a compact set $[h_m, h_M]$. We consider here a single delayed state system but this method can be easily extended to multiple delayed state system.

Lemma 3.1: Delay-dependent LPV state state-feedback existence lemma

Consider the state-delayed system (16). If there exist some symmetric positive definite matrices $Y,\ L_3,\ J_3$ and a positive scalar γ such that these LMIs hold

$$\mathcal{K}_j^T \left(\Lambda_j + T_j^T M_j T_j \right) \mathcal{K}_j < 0 \tag{18}$$

for $j = 1, 2$ and $i = 1, 2$

$$L_3, J_3 \in L_\Theta \qquad \begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0 \tag{19}$$

with

$$
\begin{array}{ll}
\Lambda_1 = -L_3 \oplus -I_1 \oplus -\gamma I_{w_2} & T_1 = \bar{D}_i \\
M_1 = L_3 \oplus I_1 \oplus \gamma^{-1} I_{z_2} & \mathcal{K}_1 = I \\
\Lambda_2 = \begin{bmatrix} Y\bar{A}^T + \bar{A}Y & Y\bar{C}_i^T \\ \star & -V_J \end{bmatrix} & V_J = J_3 \oplus I_1 \oplus \gamma I_{z_2} \\
T_2 = \begin{bmatrix} \bar{B}_i^T & \bar{D}_i^T \end{bmatrix}^T & M_2 = J_3 \oplus I_1 \oplus \gamma^{-1} s I_{w_2} \\
\mathcal{K}_2 = Ker \begin{bmatrix} \bar{B}_u^T & \bar{D}_u^{iT} \end{bmatrix} &
\end{array}
$$

then there exist a GSSF of the form (9) which stabilizes (16) with an achieved performance index lower than γ. The controller matrix can be found by solving the two following LMIs

$$
\Psi_{sf_i} + P_{sf_i}^T K Q_{sf_i} + P_{sf_i} K^T Q_{sf_i}^T < 0 \tag{20}
$$

where $K = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}$ and Ψ_{sf_i} is the matrix (6) with matrices

$$
\tilde{A} = \bar{A} \quad \tilde{B} = \begin{bmatrix} 0 \\ \bar{B}^T \end{bmatrix}^T \quad \bar{C} = \begin{bmatrix} 0 \\ \bar{C}_i \end{bmatrix} \bar{D} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{D}_i \end{bmatrix}
$$

$$
\gamma L = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix} \oplus I_1 \oplus \gamma I_{w_2}
$$

$$
\gamma L^{-1} = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}^{-1} \oplus I_1 \oplus \gamma I_{z_2}
$$

$$
P_{sf_i} = \left[\begin{array}{cc} 0 & Y^{-1}B_u \\ \hline 0 & 0 \\ \hline 0 & I \\ D_u^i & 0 \end{array}\right]^T \qquad Q_{sf_i} = \left[\begin{array}{c|cc|c} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right]
$$

IV. Delay-scheduled controllers

In the literature, two kinds of controllers are studied. Controllers with memory which contain delayed terms (for example $u(t) = K_0x(t) + K_1x(t - h(t)), \dots$) and memoryless controllers which do not contain those types of terms. Here we propose a method for the design of delay-scheduled memoryless controllers for time delay system when approximate value of the delay can be measured or estimated in real-time. This means that the delay value will appear in the controller parameter set but no delayed state is used. Moreover, we take into account in the synthesis a possible error on the delay measurement.

In this section, we provide sufficient conditions to existence of stabilizing delay-scheduled state feedback for both LTI and LPV time-delay systems.

A. Problem Formulation

We consider here that the system delay $h(t)$ can be expressed as $h(t) = h_c(t) + h_e(t)$ where $h_c(t)$ is the measured delay used in the controller and $h_e(t)$ is the measurement error such that $0 \leq |h_e(t)| < \delta_e$.

Now consider another outer representation of the system (13) where $\Theta(t) = h_n(t)$ and $\Delta = \mathcal{D}_h(\cdot) \oplus h_e(t)/\delta_e I$ (see figure 4)

$$
\begin{array}{rcl}
\dot{x} & = & \bar{A}x(t) - A_h w_1(t) + B_3 w_3(t) + B_u u(t) \\
z_0(t) & = & \bar{A}x(t) - A_h w_1(t) + B_3 w_3(t) + B_u u(t) \\
z_1(t) & = & \alpha\bar{h}\bar{A}x(t) + \alpha\delta_h w_0(t) - \alpha\bar{h}A_h w_1(t) \dots \\
 & & + \alpha\delta_e w_2(t) + \alpha\bar{h}B_3 w_3(t) + \alpha\bar{h}B_u u(t) \\
z_2(t) & = & \bar{A}x(t) - A_h w_1(t) + B_3 w_3(t) + B_u u(t) \\
z_3(t) & = & \bar{C}_3 x(t) - C_{3h} w_1(t) + D_{33} w_3(t) + D_{3u} w_3(t) \\
w_0(t) & = & h_n(t) z_0(t) \\
w_1(t) & = & \mathcal{D}_h(z_1(t)) \\
w_2(t) & = & \delta_h(t) z_2(t)
\end{array} \tag{21}
$$

where $\bar{A} = A + A_h$, $\bar{C}_3 = C_3 + C_{3h}$, $h(t) = h_c(t) + h_e(t)$, $h_c(t) = \delta_h h_n(t) + \bar{h}$, $\alpha = h_M/h_m$, δ_e is the measurement error bound, $\bar{h}$ is $\bar{h} = (h_M + h_m)/2$ is the average delay value and $\delta_h = h_M - \bar{h}$ is the delay interval deviation. h_n is defined as the common parameter for both system and controller and will be used as scheduling parameter (see figure 4).

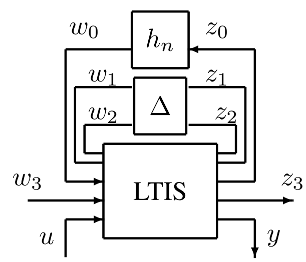

Fig. 4. Delay scheduled system

We are now able to compute either a delay-scheduled state feedback of the form (22)

$$
\dot{\eta}(t) = K(h_n(t))x(t) \tag{22}
$$

using state-space representation

$$
\bar{A} = A + A_h, \ \bar{B} = \begin{bmatrix} 0 & -A_h & 0 & B_3 \end{bmatrix}
$$

$$
\bar{C} = \begin{bmatrix} \bar{A} \\ \alpha\bar{h}\bar{A} \\ \bar{A} \\ C_3 + C_{3h} \end{bmatrix}, \ \bar{D}_u = \begin{bmatrix} B_u \\ \alpha\bar{h}B_u \\ B_u \\ D_{3u} \end{bmatrix}
$$

$$
\bar{D} = \begin{bmatrix} 0 & -A_h & 0 & B_3 \\ \alpha\delta_h I & -\alpha\bar{h}A_h & \alpha\delta_e I & \alpha\bar{h}B_3 \\ 0 & -A_h & 0 & B_3 \\ 0 & -C_{3h} & 0 & D_{33} \end{bmatrix}
$$

B. Delay-scheduled state feedback result

For that kind of controller, we must define $\bar{C}_y = I$ and $\bar{D}_{yw} = 0$. Then applying theorem 2.1 we obtain the following lemma

Lemma 4.1: Delay-scheduled state feedback existence lemma

Consider the system (21) then if there exist symmetric positive definite matrices $L_3,\ J_3,\ Y$ and a scalar $\gamma > 0$ such that the LMIs hold

$$\mathcal{K}_i^T\left(\Lambda_i + T_i^T M_i T_i\right)\mathcal{K}_i < 0,\ \text{for } i = 1, 2 \tag{23}$$

$$\begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0 \quad L_3, J_3 \in L_\Theta \tag{24}$$

with

$$\begin{array}{ll} \Lambda_1 = -L_3 \oplus -I_1 \oplus -I_2 \oplus -\gamma I_{w_3} & T_1 = \bar{D} \\ M_1 = L_3 \oplus I_1 \oplus I_{z_2} \oplus \gamma^{-1} I_{z_3} & \mathcal{K}_1 = I \\ \Lambda_2 = \begin{bmatrix} Y\bar{A}^T + \bar{A}Y & Y\bar{C}^T \\ \star & -U_J \end{bmatrix} & T_2 = \begin{bmatrix} \bar{B}^T & \bar{D}^T \end{bmatrix} \\ U_J = J_3 \oplus I_1 \oplus I_2 \oplus \gamma I_{z_3} & \mathcal{K}_2 = Ker\begin{bmatrix} \bar{B}_u^T & \bar{D}_u^T \end{bmatrix} \\ M_2 = J_3 \oplus I_1 \oplus I_2 \oplus \gamma I_{w_2} & \end{array}$$

then there exist a delay scheduled state-feedback of the form (22) which stabilizes the time-delayed system (13) with a $\mathcal{H}_\infty$ performance achievement lower than γ on the channel $w_3 \to z_3$. The controller construction method is the same as stated in the theorem 2.1.

V. Conclusion

We have presented a LPV state-feedback design for LPV time-delay systems. The present method has been extended to a LPV dynamic output feedback design in [**?**]. They have been applied for uncertain time-delay systems, after model transformation, in the delay-dependent framework. We have also developed an outer approximation of LTI state-delayed systems represented as uncertain LPV systems which can be used for delay-dependent stability criteria using a classical bounded real lemma. The latter controller uses the approximate knowledge of the delay value to schedule the controller. The error between the delay real value and the known one is taken into account into the synthesis via a robustness analysis.

Appendix

A. Proof of state feedback existence theorem - matrices L and K construction

First of all the augmented system is given by

$$\begin{bmatrix} \dot{x} \\ \hline z_c \\ z_0 \\ z_1 \\ z_2 \\ \hline x \\ w_c \end{bmatrix} = \left[\begin{array}{c|ccccc|cc} A & 0 & B_0 & B_1 & B_2 & & B_u & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 0 & I \\ C_0 & 0 & D_{00} & D_{01} & D_{02} & & D_{u0} & 0 \\ C_1 & 0 & D_{10} & D_{11} & D_{12} & & D_{u1} & 0 \\ C_2 & 0 & D_{20} & D_{21} & D_{22} & & D_{u2} & 0 \\ \hline I & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & I & 0 & 0 & 0 & & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w_c \\ w_0 \\ w_1 \\ w_2 \\ \hline u \\ z_c \end{bmatrix} \tag{25}$$

The signals w_c and z_c represent the controller parameter-scheduled channel. Let Ω be the controller state-space "static" matrices representation. Then compute the closed-loop system and inject it into the matrix inequality (6). We consider here matrices $L_w = L \oplus I_1 \oplus \gamma I_{w_2}$ and $L_z = L \oplus I_1 \oplus \gamma I_{z_2}$ where L scales the controller and system parameter-scheduled channels. It is straightforward to obtain that the inequality (6) is equivalent to the following one

$$\Psi + P^T \Omega Q + P \Omega^T Q^T < 0 \tag{26}$$

Note that this inequality is non linear in the matrix variables $X,\ \Omega,\ L_w$ and L_z^{-1}. However, it can be exactly relaxed without inducing any conservatism. Using the projection lemma we obtain the equivalent underlying problem

$$K_P^T \Psi K_P < 0 \qquad K_Q^T \Psi K_Q < 0$$

where $K_P = Ker(P)$ and $K_Q = Ker(Q)$. Then performing a Schur complement onto both inequalities leads to the LMIs used in the theorem 2.1 and the proof of the controller existence is completed.

The second part concerns the controller reconstruction. First, compute the matrices L ($J = L^{-1}$). We have computed in theorem 2.1 two matrices L_3 and J_3 which are blocks of bigger matrices defined as $L = \begin{bmatrix} L_1 & L_2 \\ \star & L_3 \end{bmatrix}$ and $J = \begin{bmatrix} J_1 & J_2 \\ \star & J_3 \end{bmatrix}$ where $LJ = I$. The algorithm is then

- Perform a SVD on $I - L_3 J_3$ to obtain the product $L_2^T J_2$ and isolate the two matrices.
- Then solve the matrix equation
$$\begin{bmatrix} 0 & I \\ J_2^T & J_3 \end{bmatrix} L = \begin{bmatrix} L_2^T & L_3 \\ 0 & I \end{bmatrix}$$
- then it is straightforward to obtain J.

Now inject the matrices describing the system (25) into the inequality (26) which is convex in Ω. Hence it is easy to solve it with convex additional constraints if necessary.

B. Proof of the $\mathcal{H}_\infty$ norm of the operator $\mathcal{D}_h(\cdot)$

Consider an input signal $w \in \mathcal{L}_2$ and an output signal $z \in \mathcal{L}_2$ such that $z(t) = \mathcal{O}(w(t))$ is a stable operator. If that operator has an $\mathcal{L}_2$ induced norm of 1 then the input and output signal must satisfy the inequality

$$\int_0^t z^T(\tau) Z z(\tau) d\tau \geq \underbrace{\int_0^t w^T(\tau) Z w(\tau) d\tau}_{\mathcal{I}}$$

for some $Z = Z^T > 0$.

The aim of this proof is first to show stability of $\mathcal{D}_h$ to prove $z \in \mathcal{L}_2 \Rightarrow w \in \mathcal{L}_2$. Secondly, one shows that the operator $\mathcal{D}_h(\cdot)$ satisfies unitary $\mathcal{H}_\infty$-norm property.

Define $h(t) \in [h_m, h_M]$, $0 < h_m < h_M < +\infty$ and

$$w(t) = \frac{h_m}{h_M} \int_{t-h(t)}^t h^{-1}(\beta) z(\beta) d\beta \in \mathcal{L}_2$$

Inject the expression of $z(t)$ into $\mathcal{I}$ and applying Jensen's inequality we obtain

$$\mathcal{I} \leq \frac{h_m^2}{h_M^2} \int_0^t h(\tau) \int_{\tau-h(\tau)}^{\tau} h^{-2}(\beta) z^T(\beta) Z z(\beta) d\beta d\tau$$

As $z \in \mathcal{L}_2$ and $h^{-1}(\tau)$ is always finite then $\int_{\tau-h(\tau)}^{\tau} h^{-2}(\beta) w^T(\beta) Z w(\beta) d\beta$ is finite. In fact, this integral term represents a band of energy from $\tau - h(\tau)$ to τ. Finally, we sum all these bands weighted by $h(\tau)$ over the interval $[0, t]$. This sum is obviously finite since $h(\cdot)$ is finite. This proves that $\mathcal{D}_h$ is a stable operator from $\mathcal{L}_2$ to $\mathcal{L}_2$.

Bounding the term $h^{-1}(\cdot)$ on the interval $[h_m, h_M]$ leads to

$$\mathcal{I} \leq \frac{1}{h_M^2} \int_0^t h(\tau) \int_{\tau-h(\tau)}^{\tau} z^T(\beta) Z z(\beta) d\beta d\tau$$

Exchange the order of integration, considering zero initial condition, to obtain

$$\mathcal{I} \leq h_M^{-2} \int_0^{t-h(t)} \left[\int h(\tau) d\tau \right] z^T(\beta) Z z(\beta) d\beta$$

but

$$\int \tau d\tau \leq h_M^2$$

and hence we have

$$\begin{aligned} \mathcal{I} &\leq \int_0^{t-h(t)} z^T(\tau) Z z(\tau) d\tau \\ &\leq \int_0^t z^T(\tau) Z z(\tau) d\tau \end{aligned}$$

which completes the proof.

A similar proof can be found in [8].

REFERENCES

[1] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers. *IEEE Transactions on Automatic Control*, 5:853–864, 1995.

[2] P. Apkarian, P. Gahinet, and G. Becker. Self scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: a design method. *Automatica*, 31(9):1251–1261, 1993.

[3] P-A. Bliman. Lyapunov equation for the stability of linear delay systems of retarded and neutral type. *IEEE Transactions on Automatic Control*, 17:327–335, 2002.

[4] C. Briat, O. Sename, and J-F. Lafay. Some $\mathcal{H}_\infty$ LFT control approaches for LPV and LTI systems with interval time-varying delays. Internal report, GIPSA-Lab, Grenoble, France, 2007.

[5] E. Fridman. Descriptor discretized lyapunov functional method: Analysis and design. *IEEE Transactions on Automatic Control*, 51:890–897, 2006.

[6] E. Fridman. Stability of systems with uncertain delays: a new 'complete' lyapunov-krasovskii functional. *IEEE Transactions on Automatic Control*, 51:885–890, 2006.

[7] F. Gouaisbaut and D. Peaucelle. Delay depependent robust stability of time delay-systems. In 5^{th} *IFAC Symposium on Robust Control Design*, Toulouse, France, 2006.

[8] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, 2003.

[9] Y. He, M. Wu, and J-H. She adn G-P. Liu. Parameter-dependent lyapunov functional for stability of time-delay systems with polytopic type uncertainties. *IEEE Transactions on Automatic Control*, 49', pages=, 2004.

[10] T. Iwasaki. LPV system analysis with quadratic separator. In *Conference on on Decision and Control*, 1998.

[11] T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions on Automatic Control*, 46:1195–1208, 2001.

[12] M. Jun and M.G. Safonov. IQC robustness analysis for time-delay systems. *Internation Journal of Robust and Nonlinear Control*, 11', pages=, 2001.

[13] Carl R. Knospe and Mardavij Roozbehani. Stability of linear systems with interval time-delay. In *IEEE American Control Conference*, Denver, Colorado, 2003.

[14] J.F. Magni. An LFT approach to robust gain scheduling. In *44th IEEE onference on Decision and Control, and the European Control Conference 2005*, pages 7971–7976, Seville, Spain, December 2005.

[15] S.-I. Niculescu. *Delay effects on stability. A robust control approach*, volume 269. Springer-Verlag: Heidelbeg, 2001.

[16] A. Packard. Gain scheduling via linear fractional transformations. *Systems and Control Letters*, 22:79–92, 1994.

[17] C. Scherer. Robust generalized $\mathcal{H}_2$ control for uncertain and LPV systems with general scalings. In *Conference on Decision and Control*, 1996.

[18] C. Scherer. A full block s-procedure with applications. In *Conference on Decision and Control*, 1997.

[19] C. Scherer. Robust mixed control and LPV control with full block scalings. *Advances in LMI Methods in Control, SIAM*, 1999.

[20] C. W. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.

[21] U. Shaked, I. Yaesh, and C.E. de Souza. Bounded real criteria for linear time-delay systems. *IEEE Transactions on Automatic Control*, 43:1016–1022, 1998.

[22] V. Suplin, E. Fridman, and U. Shaked. $\mathcal{H}_\infty$ control of linear uncertain time-delay systems - a projection approach. *IEEE Transactions on Automatic Control*, 51:680–685, 2006.

[23] K. Tan and K.M. Grigoriadis. $\mathcal{L}_2$-$\mathcal{L}_2$ and $\mathcal{L}_2$-$\mathcal{L}_\infty$ output feedback control of time-delayed LPV systems. In *Conference on on Decision and Control*, 2000.

[24] K. Tan, K.M. Grigoriadis, and F. Wu. $\mathcal{H}_\infty$ and $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain control of linear parameter varying systems with parameter varying delays. *Control Theory and Applications*, 150:509–517, 2003.

[25] F. Wu. An unified framework for LPV system analysis and control synthesis. In *Conference on Decision and Control*, 2000.

[26] F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74:745–759, 2001.

[27] F. Wu. Robust quadratic performance for time-delayed uncertain linear systems. *International Journal of Robust and Nonlinear Systems*, 13:153–172, 2003.

[28] F. Wu and K.M. Grigoriadis. LPV systems with parameter-varying time delays: analysis and control. *Automatica*, 37:221–229, 2001.

[29] F. Zhang and K.M. Grigoriadis. Delay-dependent stability analysis and $\mathcal{H}_\infty$ control for state-delayed LPV system. In *Conference on Control and Automation*, 2005.

[30] J. Zhang, C.R. Knospe, and P. Tsiotras. Stability of time-delay systems: Equivalence between lyapunov and scaled small-gain conditions. *IEEE Transactions on Automatic Control*, 46:482–486, 2001.

[31] X. Zhang, P. Tsiotras, and C. Knospe. Stability analysis of LPV time-delayed systems. *Int. Journal of Control*, 75:538–558, 2002.